Composing Quantum Instruments

Robert I. Booth[1], Dominik Leichtle[2], Alex Rice[2], and Kim Worrall[2]

[1] *University of Oxford, United Kingdom*
[2] *University of Edinburgh, United Kingdom*

Abstract

We study the composition of classically-controlled quantum instruments—the natural quantum analogue of Markov kernels. Classically, Markov kernels compose by integrating one kernel against another. Defining this composition for quantum instruments with continuous outcomes requires an integral of quantum channel-valued functions with respect to a quantum instrument. We construct this integral in the Heisenberg picture using the Okamura–Ozawa normal extension to a von Neumann tensor product. This integral recovers the expected finite formula, preserves normal complete positivity and subunitality, and provides the multiplication for a monad governing the composition of quantum instruments. As an immediate consequence, we identify the category of quantum Markov kernels as the Kleisli category of this monad.

Quantum measuring processes can be conceptually understood as consisting of two pieces of data: the probability distributions of classical outcomes given a quantum state, and the update to the quantum state conditioned on each outcome. Quantum instruments offer a mathematically elegant formulation of this viewpoint, decomposing the measuring process into a family of trace-non-increasing channels for each classical outcome that simultaneously track both data. A theory of quantum measurement with outcomes in an arbitrary measure space was formulated by Davies and Lewis [DL70], who gave a general notion of quantum instruments as quantum channel-valued measures.

This makes classically-controlled quantum instruments a natural analogue of Markov kernels in settings where classical and quantum data interact. A classical Markov kernel assigns to each input a probability measure on possible outputs. A quantum Markov kernel, in the sense considered here, assigns to each classical input a Davies-Lewis quantum instrument: the classical input controls a measuring process, whose outcomes are classical and whose effects on the quantum system are described by completely positive maps. Equivalently, such kernels may be regarded as quantum processes acting on both classical and quantum data. This begs the immediate question: how do these quantum Markov kernels compose? In particular, conditioning one quantum Markov kernel on the outcomes of another, what is the overall resulting operation?

For ordinary Markov kernels, composition is defined by integration: to compose a kernel $k : X \times \sigma_Y \to [0, 1]$ with a kernel $h : Y \times \sigma_Z \to [0, 1]$, one integrates the measurable function $y \mapsto h(y, E)$ against the measure $F \mapsto k(x, F)$. This operation is essentially the multiplication of the Giry monad [Gir82]. In the quantum case, the same idea suggests integrating a channel-valued function against a quantum instrument. However, this is no longer an ordinary scalar or even Banach-valued integral. The integrator is itself an instrument, its values are completely positive maps, and the order of composition matters.

This paper develops the integral needed for this composition. Working in the Heisenberg picture, we regard quantum channels as normal completely positive subunital (nCPSU) maps between W*-algebras. A quantum instrument is then a countably additive measure taking values in nCPSU maps. For an instrument satisfying the Okamura–Ozawa normal extension property [OO15], the instrument admits a normal completely positive extension to a W*-tensor product. This extension allows us to define a noncommutative integral of bounded measurable channel-valued functions against the instrument.

With this integral in hand, we can construct a quantum analogue of the (stateful) Giry monad, or *quantum instrument monad*, on the category of measurable spaces and measurable functions. This monad is parameterised in the sense of Atkey [Atk09], and the Kleisli category of this monad models quantum Markov kernels.

Structure In section 1, we motivate our constructions by briefly overviewing the *finite* quantum instrument monad, which is a simultaneous generalisation to the quantum setting of the finite state monad and the finite distribution monad. In section 2, we summarise some key standard results from the theory of von Neumann algebras. Section 3 constructs a *noncommutative* integral for an operator-valued function with respect to an *infinite* quantum instrument, using Okamura-Ozawa's *normal extension property*. Finally, in section 4 we introduce our quantum instrument monad.

Related work Independently of the present work, Tobias Fritz has developed a quantum instrument monad in the Schrödinger picture. The technical details differ—in particular, the construction of the quantum instrument integral—but the resulting constructions appear to be closely related. We refer the reader to [Fri26] for this complementary perspective.

1 Motivation

A quantum measurement process is commonly described by a *quantum instrument*: a finite family of completely positive maps $\{\Phi_x\}$ such that $\sum_x \Phi_x$ is trace-preserving. The quantity $\operatorname{tr}[\Phi_x(\rho)]$ gives the probability of obtaining outcome x in state ρ, while the normalized state $\Phi_x(\rho)/\operatorname{tr}[\Phi_x(\rho)]$ describes the corresponding post-measurement state. Thus, the family $\{\Phi_x\}$ simultaneously encodes both the measurement statistics and the state update induced by the measurement.

Example 1. Let $\{P_i\}$ be an orthogonal family of projectors such that $\sum_i P_i = 1$, then the quantum instrument associated with this measurement is given by the family $\Phi_i(\rho) = P_i \rho P_i$.

For infinite-dimensional quantum systems, it will be convenient to use the Heisenberg picture. Thus a measurement process from a system with Hilbert space $\mathcal{H}$ to one with Hilbert space $\mathcal{K}$, with outcomes in a finite or countable set X, is described by a family of completely positive maps

$$\Phi_x : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H}) \tag{1}$$

such that $\sum_x \Phi_x$ is unital. Given an initial state ω on $\mathcal{B}(\mathcal{H})$, the probability of obtaining outcome x is

$$p_x := \omega(\Phi_x(1_{\mathcal{K}})). \tag{2}$$

Note that the positivity of Φ_x guarantees $p_x \geq 0$ while the normalisation (unitality) of $\sum_x \Phi_x$ implies that $\sum_x p_x = 1$. Conditioned on this outcome, the resulting state on $\mathcal{B}(\mathcal{K})$ is

$$\omega_x(A) = \frac{\omega(\Phi_x(A))}{\omega(\Phi_x(1_{\mathcal{K}}))}, \qquad A \in \mathcal{B}(\mathcal{K}), \tag{3}$$

provided the denominator is nonzero.

The preceding discussion suggests that quantum instruments possess a rich compositional structure. A measurement process may be followed by a second measurement, and the choice of the second measurement may itself depend on the outcome of the first. Consequently, one would like a mathematical formalism in which measurement procedures can be assembled into more complicated measurement procedures.

Two distinct mechanisms are at work. First, a measurement transforms the quantum state, so that the output state of one instrument becomes the input state of the next. Second, a measurement produces outcomes with specified probabilities, introducing a form of probabilistic branching.

These two phenomena are familiar from other contexts. State evolution is captured categorically by the state monad and its parameterised variants [Mog91; Atk09], while probabilistic branching is captured by probability monads such as the finite distribution monad [Fri09]. Quantum instruments combine both features in a single object: they describe probabilistic processes which also transform an underlying quantum state.

Moreover, both aspects participate in the composition of measurement processes. States evolve sequentially through successive instruments, while measurement outcomes determine how the process branches and which subsequent instrument is applied. The finite-dimensional instrument construction considered below provides a mathematical framework in which these two forms of composition coexist. The resulting structure carries the form of a parameterised monad, simultaneously extending the parameterised state monad and the finite distribution monad. This finite-dimensional setting serves as a useful toy model for the more general quantum instrument monad developed in the remainder of the paper.

We give the formal definitions of monads and parameterised monads in appendix A, and focus here on motivating the monads we are interested in for modelling quantum Markov kernels.

1.1 The parameterised state monad

Suppose a physical system has state space S. A *perturbative observation* of the system naturally modelled by a function

$$S \longrightarrow X \times S. \tag{4}$$

which returns an observable outcome in some set X, while possible altering the state of the system. Equivalently, the collection of all perturbative observations of S with outcomes in X is

$$M^S X \coloneqq \mathsf{Set}(S, X \times S). \tag{5}$$

In general, when performing a sequence of observations of the system S, one might vary the choice of observations as a function of the outcome of previous observations. This endows the assignment $X \mapsto M^S X$ with a non-trivial compositional structure which is naturally captured by the *state monad*.

In many situations, however, the type of the state itself need not remain fixed. An observation may transform states of one type into states of another, as is common for quantum channels. This leads naturally to Atkey's *parameterised state monad* [Atk09], defined by

$$M^{S,T} X \coloneqq \mathsf{Set}(S, X \times T). \tag{6}$$

Here an observation begins with a state in S, produces an outcome in X, and leaves the system in a state belonging to T. As before, observations can be composed whenever the output state space of one agrees with the input state space of the next. This composition equips the family $\{M^{S,T}\}$ with the structure of a parameterised monad. Operationally, the multiplication corresponds to performing an observation whose outcome determines a subsequent observation.

The parameterised state monad provides a natural model of stateful physical processes whose state space may change. This is precisely the situation encountered for quantum instruments, where measurements may transform states between different Hilbert spaces.

1.2 The finite distribution monad

Just as the state monad captures stateful processes, probability monads capture probabilistic branching. The simplest example is the finite distribution monad. Given a set X, let $\mathcal{D}(X)$ denote the set of finitely supported probability distributions on X. A function

$$f : X \to \mathcal{D}(Y) \tag{7}$$

may be interpreted as a probabilistic process: given an input x, the process returns an output distributed according to the probability distribution $f(x)$.

The unit of the monad sends an element to the corresponding Dirac distribution,

$$\eta_X(x) = \delta_x, \tag{8}$$

while the multiplication

$$\begin{aligned} \mu_X : \mathcal{D}(\mathcal{D}(X)) &\longrightarrow \mathcal{D}(X) \\ p &\longmapsto \sum_{d \in \mathcal{D}(X)} p_d \cdot d \end{aligned} \tag{9}$$

averages a probability distribution over probability distributions into a single probability distribution. Operationally, this corresponds to performing a probabilistic choice whose outcomes are themselves probabilistic processes.

This monad provides a categorical model of probabilistic branching. In particular, sequential composition of probabilistic processes is encoded by the monad multiplication, exactly as sequential composition of stateful computations is encoded by the state monad.

The finite distribution monad admits a natural extension from sets to measurable spaces called the *Giry monad* [Gir82], in which finitely supported distributions are replaced by probability measures. The Giry monad plays a central role in categorical probability theory and will provide one of the principal ingredients in our construction of the quantum instrument monad.

1.3 The finite quantum instrument monad

The preceding discussion suggests that quantum instruments should admit a parameterised monadic structure. We now describe the finite version of this construction.

Like the parameterised state monad, the underlying state space is allowed to vary. In the quantum setting the state spaces are Hilbert spaces, and the state transformations are completely positive maps. Like the finite distribution monad, measurements branch over a set of classical outcomes. Accordingly, for Hilbert spaces $\mathcal{H}, \mathcal{K}$ and an outcome set X, let $Q^{\mathcal{H},\mathcal{K}}X$ be the set of finitely-supported quantum instruments, that is $\{\Phi_x : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})\}_{x \in X}$, such that $\sum_{x \in X} \Phi_x$ is unital.

The units are analogous to the Dirac distribution from the finite distribution monad. For any Hilbert space $\mathcal{H}$ and $x \in X$, define a *Dirac instrument* as

$$\left(\delta_x^{\mathcal{H}}\right)_y = \begin{cases} 1_{\mathcal{H}} \text{ if } x = y; \\ 0 \text{ otherwise.} \end{cases} \tag{10}$$

where $1_{\mathcal{H}}$ is the identity channel. The units of the monad are given by $\eta_X^{\mathcal{H}}(x) = \delta_x^{\mathcal{H}}$.

An element of $Q^{\mathcal{H},\mathcal{J}}Q^{\mathcal{J},\mathcal{K}}X$ may be viewed as a two-stage measurement procedure: one first performs a measurement on $\mathcal{H}$, which transforms the quantum system to the Hilbert space $\mathcal{J}$, and whose classical outcomes also determine a second instrument on $\mathcal{J}$. The multiplication simply regards this adaptive two-stage procedure as a single instrument from $\mathcal{H}$ to $\mathcal{K}$. Explicitly,

$$\left(\mu_X^{\mathcal{H},\mathcal{J},\mathcal{K}}(p)\right)_x = \sum_{q \in Q^{\mathcal{J},\mathcal{K}}X} p_q \circ q_x. \tag{11}$$

Here $p_q : \mathcal{B}(\mathcal{J}) \to \mathcal{B}(\mathcal{H})$ is the operation associated with the outer outcome q, while $q_x : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{J})$ is the operation associated with outcome x of the inner instrument q. Note that only finitely many of the summands can be non-zero.

This combines the two constructions discussed above. As in the finite distribution monad, the outer measurement averages over the possible intermediate outcomes. As in the parameterised state monad, the intermediate quantum state produced by the first instrument is passed to the second.

Theorem 2. *The constructions above endow the family $Q^{\mathcal{H},\mathcal{K}}$ with the structure of a parameterised monad.*

Rather than verifying the axioms directly in this finite setting, we defer the proof to the general measurable construction developed below, of which the present monad is a special case.

2 Elements of W*-algebras and quantum channels

Definition 3. *A* W*-algebra *is a Banach *-algebra $\mathcal{A}$ which is the topological dual of a Banach algebra $\mathcal{A}_*$ called its* predual*. The weak-* topology induced on $\mathcal{A}$ by the functionals in $\mathcal{A}_* \subseteq (\mathcal{A}_*)^{**} = \mathcal{A}^*$ is called the* ultraweak *topology.*

W*-algebras come with sufficient structure to define a notion of positivity, equipping them with a natural order structure. In the following, we review only some of their properties that are important for our purposes. For a more detailed discussion, we refer to [Sak12].

Definition 4. *Let $\mathcal{A}, \mathcal{B}$ be W*-algebras, then a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is*

- normal *if it is ultraweakly continuous;*
- unital *if $\Phi(1_\mathcal{A}) = 1_\mathcal{B}$;*
- subunital *if $\Phi(1_\mathcal{A}) \leqslant 1_\mathcal{B}$;*
- positive *if $\Phi(a) \geqslant 0$ for every $a \geqslant 0$.*

Proposition 5 ([Cho16])**.** *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be positive. Then Φ is normal if and only if it preserves the supremum of norm-bounded monotone nets in $\mathcal{A}$.*

Consider W*-algebras $\mathcal{A}, \mathcal{B}$, and a linear map $\Phi : \mathcal{A} \to \mathcal{B}$. The *matrix amplifications* of Φ are the linear maps defined for each $n \in \mathbb{N}$ by:

$$\begin{aligned} \Phi^{(n)} : M_n(\mathcal{A}) &\longrightarrow M_n(\mathcal{B}) \\ [a_{jk}] &\longmapsto [\Phi(a_{jk})]. \end{aligned} \tag{12}$$

Definition 6. *We say that Φ is* completely bounded *if each $\Phi^{(n)}$ is bounded and* completely positive *if each $\Phi^{(n)}$ is positive.*

Proposition 7 ([Cho16])**.** *There is a symmetric monoidal category $\mathsf{W}^*_{\mathsf{CPSU}}$ where*

- objects *are W*-algebras;*

- arrows *are normal completely positive subunital linear maps;*
- *the* monoidal structure *is given by the spatial von Neumann tensor product, or W*-tensor,* $\bar{\otimes}$ *(see [Sak12, Definition 1.22.10] for a full definition).*

We impose throughout that all W*-algebras have a separable predual. The key consequence of this assumption is lemma 9, which informally says that all quantum instruments can be thought of as densities with respect to a real measure.

3 The integral theory of quantum instruments

Following the discussion of the discrete-outcome case in section 1, we now move towards quantum instruments with continuous outcomes by revisiting Davies' and Lewis' definition of quantum instruments on measurable spaces which generalise both finite-outcome quantum instruments and classical distributions in the form of probability measures.

Definition 8 (Quantum instrument [DL70])**.** *Let $\mathcal{A}, \mathcal{B}$ be W*-algebras and (X, σ_X) a measurable space, then a* quantum instrument $\mathcal{A} \to_X \mathcal{B}$ *is a mapping $q : \sigma_X \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B})$ which is*

- strict*: $q(\varnothing) = 0_{\mathcal{A},\mathcal{B}}$;*
- normalised*: $q(X)(1_{\mathcal{A}}) = 1_{\mathcal{B}}$;*
- ultraweakly countably additive*: for any countable collection $\{E_j\} \subseteq \sigma_X$ of disjoint sets, $a \in \mathcal{A}$, and $\phi \in \mathcal{B}_*$, $\langle \phi, q(\bigcup_j E_j)(a)\rangle = \sum_j \langle \phi, q(E_j)(a)\rangle$.*

In particular, for any quantum instrument $q : \mathcal{A} \to_X \mathcal{B}$ and any (respectively, positive unital) $\phi \in \mathcal{B}_*$, the mapping $q_\phi : \sigma_X \to \mathbb{R}_+, \ E \mapsto \langle \phi, q(E)(1_{\mathcal{A}})\rangle$ defines a complex (respectively, probability) measure on X.

Lemma 9. *For any quantum instrument $q : \mathcal{A} \to_X \mathcal{B}$, there is a finite real measure ν on X which dominates q. Explicitly, $q(E) = 0_{\mathcal{A},\mathcal{B}}$ whenever $\nu(E) = 0$.*

Proof. Since $\mathcal{B}_*$ is separable (by assumption), there is a faithful positive state $s \in \mathcal{B}_*$. Define a finite real measure $\nu \coloneqq E \mapsto \langle s, q(E)(1_{\mathcal{A}})\rangle$, then if $\nu(E) = \langle s, q(E)(1)\rangle = 0$, we must have $q(E)(1) = 0$, which since $q(E)$ is completely positive implies $q(E) = 0$. □

Since quantum instruments are in effect CPSU-valued measures, we would like to define a *noncommutative* integral of CPSU-valued functions Φ w.r.t. a quantum instrument q

$$\int_X q(\mathrm{d}x) \circ \Phi(x), \tag{13}$$

which extends the case where q takes only finitely many values and the integral reduces to a straightforward sum. This integral suggests the existence of "left" and "right" orderings, or pre- and post-composition by Φ. We focus on the pre-composition integral since, as we shall see in the sequel, this is the ordering that corresponds to the physically relevant composition of quantum stochastic processes. In particular, one can already see in the finite case that only the pre-composition integral guarantees subunitality of the resulting CP map.

We construct this integral via the well-known connection between operator-valued measurable functions, measurable fields of operators, and tensor products. Many of these statements are either known, expected or folklore, but we provide proofs for those statements we could not find verbatim in the literature for the sake of completeness.

Proposition 10. *Let (X, σ_X) be a measurable space, ν a σ-finite measure on X, $\mathcal{A}$ a W*-algebra, and $\Phi : X \to \mathcal{A}$ a bounded ultraweakly measurable map. Then, Φ determines a unique element $[\Phi]_\nu \in L^\infty(\nu) \bar{\otimes} \mathcal{A}$ characterised by*

$$\langle f \otimes \phi, [\Phi]_\nu \rangle = \int_X f(x) \cdot \langle \phi, \Phi(x) \rangle \ \nu(\mathrm{d}x), \quad \textit{where} \quad f \in L^1(\nu),\ \phi \in \mathcal{A}_*.$$

Note that Proposition 10 holds in the slightly more general case in which Φ is only essentially bounded, up to null-sets of ν, rather than bounded everywhere. Since we are interested in the case in which the integral of Φ can be defined along *any* quantum instrument, we restrict to the latter case in the following.

Proof. For each $\phi \in \mathcal{A}_*$, the map $X \to \mathbb{C} : x \mapsto \langle \phi, \Phi(x) \rangle$ is bounded and measurable. It therefore determines a bounded bilinear map

$$\begin{aligned} L^1(\nu) \times \mathcal{A}_* &\longrightarrow \mathbb{C} \\ (f, \phi) &\longmapsto \int_X f(x) \cdot \langle \phi, \Phi(x) \rangle \ \nu(\mathrm{d}x), \end{aligned} \tag{14}$$

which, by the universal property of the projective tensor product [Hel06, Theorem 2.7.4], extends uniquely to a bounded linear map $L^1(\nu) \hat{\otimes} \mathcal{A}_* \to \mathbb{C}$. Recalling the identifications $L^\infty(\nu) \bar{\otimes} \mathcal{A} \cong L^1(\nu, \mathcal{A}_*)^*$ [Sak12, Theorem 1.22.13] and $L^1(\nu, \mathcal{A}_*) \cong L^1(\nu) \hat{\otimes} \mathcal{A}_*$ [Sak12, Theorem 1.22.12] this yields an element $[\Phi]_\nu \in L^\infty(\nu) \bar{\otimes} \mathcal{A}$ modulo null-sets of ν, as claimed. □

As a result, any reasonable integral with respect to a quantum instrument $q : \mathcal{A} \to_X \mathcal{B}$ should be given by a map $\overline{q} : L^\infty(\nu) \bar{\otimes} \mathcal{A} \to \mathcal{B}$. Simple functions $s := \sum_k a_k \cdot 1_{E_k}$, where $a_k \in \mathcal{A}$ and 1_{E_k} is the characteristic function of the measurable set $E_k \in \sigma_X$, get mapped to $[s]_\nu = \sum_k 1_{E_k} \otimes a_k$. In other words, approximating arbitrary integrands by measurable functions in the ultraweak sense, we should expect this function $\overline{q}$ to be a normal completely positive unital linear map.

Okamura and Ozawa [OO15] show that every quantum instrument $\mathcal{A} \to_X \mathcal{B}$ is equivalent to a unital completely positive map $L^\infty(\nu) \otimes_{\mathrm{bin}} \mathcal{A} \to \mathcal{B}$, where $\otimes_{\mathrm{bin}}$ is the *C*-algebraic binormal tensor*. In order to respect the measurable i.e. W*-algebraic structure we are interested in, we would like this linear map to extend to a *normal* map $L^\infty(\nu) \bar{\otimes} \mathcal{A} \to \mathcal{B}$.

Definition 11 (Normal extension property [OO15])**.** *A quantum instrument $q : \mathcal{A} \to_X \mathcal{B}$ has the* normal extension property *(NEP) if there exists a finite measure ν_ρ on X and a normal completely positive unital map $\overline{q} : L^\infty(\nu_\rho) \bar{\otimes} \mathcal{A} \to \mathcal{B}$ such that $q(E)(a) = \overline{q}(1_E \otimes a)$ for all $E \in \sigma_X$ and $a \in \mathcal{A}$.*

In order to simplify notation, for any bounded measurable $\Phi : X \to \mathcal{A}$, we write $[\Phi]_q := [\Phi]_{\nu_q}$ the measurable field obtained from proposition 10. If q has the NEP, we can straightforwardly define an integral of Φ with respect to q as

$$\int_X q(\mathrm{d}x) \circ \Phi(x) := \overline{q}([\Phi]_q). \tag{15}$$

Unfortunately, according to Okamura and Ozawa, "usually it is not easy to check whether a given CP instrument has the NEP or not". They prove that quantum instruments whose codomain is an *atomic* W*-algebra always have the NEP. However, outside this case, simple counterexamples already exist, such as sharp measurements on $L^\infty([0,1])$.

Definition 12. *A map $\Phi : X \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B})$ is* measurable *if for all $\phi \in \mathcal{B}_*$ and $a \in \mathcal{A}$, the mapping $x \mapsto \langle \phi, \Phi(x)(a) \rangle$ is Borel measurable.*

In particular, for any bounded measurable $\Phi : X \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B})$, and each $a \in \mathcal{A}$, the map $\Phi(-)(a) : X \to \mathcal{B} : x \mapsto \Phi(x)(a)$ is bounded and ultraweakly measurable, so we can define an integral along an NEP quantum instrument $q : \mathcal{B} \to_X \mathcal{C}$ pointwise as

$$\left(\int_X q(\mathrm{d}x) \circ \Phi\right)(a) \coloneqq \overline{q}([\Phi(-)(a)]_q). \tag{16}$$

Proposition 13. *For any bounded measurable map $\Phi : X \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B})$ and NEP quantum instrument $q : \mathcal{B} \to_X \mathcal{C}$, $\int_X q(\mathrm{d}x) \circ \Phi$ is a normal completely positive subunital map $\mathcal{A} \to \mathcal{C}$.*

Proof. By assumption, we have an integral $\overline{q} : L^\infty(\nu_q) \mathbin{\bar{\otimes}} \mathcal{B} \to \mathcal{C}$ and by proposition 10 a map

$$\begin{aligned} \overline{\Phi} : \mathcal{A} &\longrightarrow L^\infty(\nu_q) \mathbin{\bar{\otimes}} \mathcal{B} \\ a &\longmapsto [\Phi(-)(a)]_q \end{aligned} \tag{17}$$

which is uniquely defined by the property that

$$\left\langle f \otimes \phi, \overline{\Phi}(a)\right\rangle = \int_X f(x) \cdot \langle \phi, \Phi(x)(a)\rangle \; \nu_q(\mathrm{d}x), \quad \text{where} \quad f \in L^1(\nu_q),\ \phi \in \mathcal{B}_*, \tag{18}$$

so $\overline{\Phi}$ is linear and bounded.

▷ $\overline{\Phi}$ *is subunital.* For any positive $f \in L^1(\nu_q)$ and positive $\phi \in \mathcal{B}_*$,

$$\begin{aligned} \left\langle f \otimes \phi, \overline{\Phi}(1_{\mathcal{A}})\right\rangle &= \int_X f(x) \cdot \langle \phi, \Phi(x)(1_{\mathcal{A}})\rangle \; \nu_q(\mathrm{d}x), && (19)\\ &\leqslant \langle \phi, 1_{\mathcal{B}}\rangle \cdot \int_X f(x) \; \nu_q(\mathrm{d}x) && (20)\\ &= \langle f \otimes \phi, 1_X \otimes 1_{\mathcal{B}}\rangle, && (21) \end{aligned}$$

so $\overline{\Phi}$ is subunital.

▷ $\overline{\Phi}$ *is completely positive.* Consider the matrix amplification $\overline{\Phi}^{(n)}$, which is uniquely defined by $\overline{\Phi}^{(n)}([a_{jk}]) = [[\Phi(-)(a)]_q]_{jk}$, so that in particular, for any $f \in L^1(\nu_q)$ and $[\phi_{jk}] \in M_n(\mathcal{B}_*)$,

$$\begin{aligned} \left\langle f \otimes [\phi_{jk}], \overline{\Phi}^{(n)}([a_{uv}])\right\rangle &= \langle f \otimes [\phi_{jk}], [[\Phi(-)(a)]_q]_{jk})\rangle && (22)\\ &= \int_X f(x) \cdot \langle [\phi_{jk}], [\Phi(x)(a)]_{jk})\rangle \; \nu_q(\mathrm{d}x) && (23)\\ &= \int_X f(x) \cdot \langle [\phi_{jk}], \Phi(x)^{(n)}([a_{uv}])\rangle \; \nu_q(\mathrm{d}x). && (24) \end{aligned}$$

Recalling the identification $M_n(L^\infty(\nu_q) \mathbin{\bar{\otimes}} \mathcal{B})_* \cong L^1(\nu_q) \mathbin{\hat{\otimes}} M_n(\mathcal{B}_*)$, for any positive $f \in L^1(\nu_q)$, positive $[a_{uv}] \in M_n(\mathcal{A})$ and positive $[\phi_{jk}] \in M_n(\mathcal{B}_*)$,

$$\left\langle f \otimes [\phi_{jk}], \overline{\Phi}^{(n)}([a_{uv}])\right\rangle = \int_X f(x) \cdot \langle [\phi_{jk}], \Phi(x)^{(n)}([a_{uv}])\rangle \; \nu_q(\mathrm{d}x), \tag{25}$$

which is positive since $\Phi(x)^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is positive by assumption.

▷ $\overline{\Phi}$ *is normal.* Consider a positive monotone net $a_k \to a$ in $\mathcal{A}$, then by the monotone convergence theorem, for any positive $f \in L^1(\nu_q)$ and $\phi \in \mathcal{B}_*$,

$$\left\langle f \otimes \phi, \overline{\Phi}(a_k) \right\rangle = \int_X f(x) \cdot \langle \phi, \Phi(x)(a_k) \rangle \; \nu_q(\mathrm{d}x) \tag{26}$$

$$\to \int_X f(x) \cdot \langle \phi, \Phi(x)(a) \rangle \; \nu_q(\mathrm{d}x) \tag{27}$$

$$= \left\langle f \otimes \phi, \overline{\Phi}(a) \right\rangle , \tag{28}$$

and this extends to evaluation with respect to any element of $L^1(\nu_q) \hat{\otimes} \mathcal{B}_*$ by norm density of algebraic tensors. Therefore, $\overline{\Phi}(a_k) \to \overline{\Phi}(a)$ and $\overline{\Phi}$ is normal by Proposition 5.

Finally, as the composition of normal completely positive subunital maps, $\int_X q(\mathrm{d}x) \circ \Phi(x) = \overline{q} \circ \overline{\Phi}$ is itself a normal completely positive subunital map. □

4 The quantum instrument monad

4.1 The parameterised functor

The *quantum instrument monad* is defined on the $\mathsf{W}^*_{\mathsf{CPSU}}$-indexed functor $Q^{\mathcal{A},\mathcal{B}}$ on $\mathsf{Measurable}$, which maps $(X, \sigma_X) \in \mathsf{Measurable}$ to the set of all NEP quantum instruments $\mathcal{A} \to_X \mathcal{B}$ over (X, σ_X). We equip this set with the coarsest σ-algebra that makes the evaluation mappings

$$\begin{aligned} \mathrm{ev}_{U,a,b} : Q^{\mathcal{A},\mathcal{B}}(X, \sigma_X) &\longrightarrow \mathbb{C} \\ q &\longmapsto \langle a, q(U)(b) \rangle \end{aligned} \tag{29}$$

Borel-measurable for each $U \in \sigma_X$, $a \in \mathcal{A}_*$, and $b \in \mathcal{B}$.

Its action on morphisms is given in the expected way by *pushforward*: for any $f \in \mathsf{Measurable}(X, Y)$ we define a map $Q^{\mathcal{A},\mathcal{B}}(f) : Q^{\mathcal{A},\mathcal{B}}(X) \to Q^{\mathcal{A},\mathcal{B}}(Y)$, given for any quantum instrument $q \in Q^{\mathcal{A},\mathcal{B}}(X)$ and $E \in \sigma_Y$ by $Q^{\mathcal{A},\mathcal{B}}(f)(q)(E) := q(f^{-1}(E))$. Since

$$\mathrm{ev}_{B,a,b} \circ Q^{\mathcal{A},\mathcal{B}}(f)(q) = \langle a, q(f^{-1}(B))(b) \rangle = \mathrm{ev}_{f^{-1}(B),a,b}(q) \tag{30}$$

for any $B \in \sigma_Y$, $a \in \mathcal{A}_*$, $b \in \mathcal{B}$ and $q \in Q^{\mathcal{A},\mathcal{B}}(X, \sigma_X)$, $Q^{\mathcal{A},\mathcal{B}}(f)$ is measurable.

4.2 The monad structure

Following the discussion of section 1, we want to show that $Q^{\mathcal{A},\mathcal{B}}$ admits a parameterised monad structure. By analogy with the Giry monad, for any $\mathcal{A} \in \mathsf{W}^*_{\mathsf{CPSU}}$ and $x \in X$, define a *Dirac instrument* by

$$\begin{aligned} \delta^{\mathcal{A}}_x : \sigma_X &\longrightarrow \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{A}) \\ U &\longmapsto \begin{cases} 1_{\mathcal{A}} \text{ if } x \in U; \\ 0 \text{ otherwise.} \end{cases} \end{aligned} \tag{31}$$

The units of the monad are given by $\eta^{\mathcal{A}}_X : X \to Q^{\mathcal{A},\mathcal{A}}(X) : x \mapsto \delta^{\mathcal{A}}_x$.

Lemma 14. *For any $x \in X$, $\eta^{\mathcal{A}}_X(x) = \delta^{\mathcal{A}}_x$ has the NEP. Explicitly, for any bounded ultraweakly measurable map $\Phi : X \to \mathcal{A}$, $\overline{\delta^{\mathcal{A}}_x}([\Phi]_{\delta^{\mathcal{A}}_x}) = \Phi(x)$.*

Proof. Denote δ_x the Dirac measure at $x \in X$, then Dirac instruments have the NEP, with $\overline{\delta^{\mathcal{A}}_x} : L^\infty(\delta_x) \mathbin{\bar{\otimes}} \mathcal{A} \to \mathcal{A}$ given on algebraic tensors by $\overline{\delta^{\mathcal{A}}_x}(f \otimes a) = f(x) \cdot a$.

It's straightforward that this mapping is ultraweakly continuous and completely positive on algebraic tensors, hence extends uniquely to a normal CPU map as claimed. □

Lemma 15. *$\eta^{\mathcal{A}}_X$ is measurable.*

Proof. We have $\mathrm{ev}_{U,a,b} \circ \eta_X^{\mathcal{A}}(x) = \mathrm{ev}_{U,a,b}(\delta_x) = \langle a, \delta_x(U)(b)\rangle = \langle a, b\rangle \cdot 1_U(x)$ which is measurable since indicator functions of measurable sets are always measurable. ☐

Lemma 16. *The family $(\eta_X^{\mathcal{A}})_X$ describes a natural transformation $1_X \mapsto Q^{\mathcal{A},\mathcal{A}}$.*

Proof. This is entirely analogous to the Giry case. ☐

By analogy with the finite case and probability monads, we define the monad multiplication as the integral

$$\begin{aligned} \mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}} : Q^{\mathcal{B},\mathcal{C}} Q^{\mathcal{A},\mathcal{B}} X &\longrightarrow Q^{\mathcal{A},\mathcal{C}} X, \\ p &\longmapsto \left(E \mapsto \int_{Q^{\mathcal{A},\mathcal{B}}(X)} p(\mathrm{d}q) \circ q(E) \right). \end{aligned} \tag{32}$$

More explicitly, $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)(E)(a) = \tilde{p}([q \mapsto q(E)(a)]_p)$.

Lemma 17. *For any $p \in Q^{\mathcal{B},\mathcal{C}} Q^{\mathcal{A},\mathcal{B}} X$, $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)$ is a quantum instrument $\mathcal{A} \to_X \mathcal{C}$.*

Proof. ▷ *$q \mapsto q(E)(a)$ is bounded.* For any $q \in Q^{\mathcal{A},\mathcal{B}} X$, $E \in \sigma_X$ and $a \in \mathcal{A}$, we have that $q(E)(a) \leqslant \|a\| \cdot q(E)(1_{\mathcal{A}}) \leqslant \|a\| 1_{\mathcal{B}}$, so that $\|q(E)(a)\| \leqslant \|a\|$.

▷ *$q \mapsto q(E)(a)$ is ultraweakly measurable.* For any $E \in \sigma_X$, $a \in \mathcal{A}$, and $b \in \mathcal{B}_*$, the mapping $Q^{\mathcal{A},\mathcal{B}} X \to \mathbb{C} : q \mapsto \langle b, q(E)(a)\rangle$ is measurable, by definition of the measurable structure on $Q^{\mathcal{A},\mathcal{B}} X$, so that the mapping $Q^{\mathcal{A},\mathcal{B}} X \to \mathcal{B} : q \mapsto q(E)(a)$ is also measurable.

Therefore, by proposition 13, the integral

$$\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)(E)(a) := \int_{Q^{\mathcal{A},\mathcal{B}}(X)} p(\mathrm{d}q) \circ q(E)(a) = \overline{p}([q \mapsto q(E)(a)]_p) \tag{33}$$

is well-defined, and $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)(E)$ is normal, subunital and completely-positive.

▷ *Countable additivity and strictness.* Let $(E_k)_{k \in \mathbb{N}} \subseteq \sigma_Y$ be pairwise disjoint and put $E = \bigcup_k E_k$, then since any $q \in Q^{\mathcal{A},\mathcal{B}} X$ is subadditivite, $q(E) = \sum_k q(E_k)$. Then by linearity of $\overline{p}$ and of the mapping $\Phi \to [\Phi]_p$,

$$\begin{aligned} \int_{Q^{\mathcal{A},\mathcal{B}} X} p(\mathrm{d}q) \circ q(E) &= \overline{p}([q \mapsto q(E)]_p) && (34) \\ &= \overline{p}([q \mapsto \sum_k q(E_k)]_p) && (35) \\ &= \sum_k \overline{p}([q \mapsto q(E_k)]_p) && (36) \\ &= \sum_k \int_{Q^{\mathcal{A},\mathcal{B}} X} p(\mathrm{d}q) \circ q(E_k), && (37) \end{aligned}$$

with ultraweak convergence of the sum. Since for all $q \in Q^{\mathcal{A},\mathcal{B}} X$ we have $q(\varnothing) = 0$, we get $\int_{Q^{\mathcal{A},\mathcal{B}} X} p(\mathrm{d}q) \circ q(\varnothing) = 0$.

▷ *Normalisation.* This follows almost immediately since

$$\int_{Q^{\mathcal{A},\mathcal{B}} X} p(\mathrm{d}q) \circ q(X)(1_{\mathcal{A}}) = \overline{p}([q \mapsto q(X)(1_{\mathcal{A}})]_p) = \overline{p}([q \mapsto 1_{\mathcal{B}}]_p) = \overline{p}(1_{Q^{\mathcal{A},\mathcal{B}} X} \otimes 1_{\mathcal{B}}) = 1_{\mathcal{C}}. \tag{38}$$

Altogether, we have shown that the assignment

$$\sigma_Y \longrightarrow \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A},\mathcal{C}) : E \longmapsto \int_{Q^{\mathcal{A},\mathcal{B}}X} \mu(\mathrm{d}q) \circ q(E) \tag{39}$$

yields a strict, normalised, countably additive map $\sigma_Y \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A},\mathcal{C})$. Hence it is a quantum instrument $\mathcal{A} \to \mathcal{C}$ on (Y, σ_Y), as claimed. □

Lemma 18. *For any $p \in Q^{\mathcal{B},\mathcal{C}}Q^{\mathcal{A},\mathcal{B}}X$, $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)$ has the NEP. Explicitly, for any bounded ultraweakly measurable $\Phi : Q^{\mathcal{A},\mathcal{B}}X \to \mathcal{B}$,*

$$\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)}([\Phi]_{\mu(p)}) = \overline{p}\big([q \mapsto \overline{q}([\Phi]_q)]_p\big).$$

where $[\Phi]_{\mu(p)}$ is shorthand for the measurable field of Φ modulo the null-sets of $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)$.

Proof. By lemma 17, $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)$ is a quantum instrument, hence there is a measure $\nu_{\mu(p)}$ on X that dominates it. By assumption, p has the NEP, so there is a measure ν_p on $Q^{\mathcal{A},\mathcal{B}}X$ and a normal CPU map $\overline{p} : L^\infty(\nu_p) \mathbin{\bar\otimes} \mathcal{B} \to \mathcal{C}$ where $p(E)(a) = \overline{p}(1_E \otimes a)$.

For any bounded ultraweakly measurable map $\Phi : Q^{\mathcal{A},\mathcal{B}}X \to \mathcal{A}$, consider the mapping $Q^{\mathcal{A},\mathcal{B}}X \to \mathcal{B} : q \mapsto \overline{q}([\Phi]_q)$. This mapping is bounded by $\|z\|$ and measurable since $\langle \phi, \overline{\mathrm{ev}}_\Phi(q)\rangle = \langle \phi, \overline{q}([\Phi]_q)\rangle$ is the composition of the measurable maps $q \mapsto \overline{q}$ and $\mathrm{ev}_{\phi,[\Phi]_q}$, so we can define

$$K(\Phi) = \overline{p}([q \mapsto \overline{\mathrm{ev}}_\Phi(q)]_p) = \overline{p}([q \mapsto \overline{q}([\Phi]_q)]_p) \in \mathcal{C} \tag{40}$$

so that for simple functions $s_{E,a}(x) = 1_E(x) \cdot a$,

$$K(s_{E,a}) = \overline{p}([q \mapsto q(E)(a)]_p) = \mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)(E)(a). \tag{41}$$

It remains to check that K depends only on the class $[\Phi]_{\mu(p)}$. Suppose first that Φ is positive and vanishes outside a $\nu_{\mu(p)}$-null set N. Then

$$0 \leqslant [\Phi]_q \leqslant \|\Phi\| \cdot [1_N \otimes 1_A]_q \tag{42}$$

for each q, hence

$$0 \leqslant \overline{q}([\Phi]_q) \leqslant \|\Phi\| \cdot q(N)(1_A). \tag{43}$$

Applying $\overline{p}$ gives

$$0 \leqslant K(\Phi) \leqslant \|\Phi\| \cdot \overline{p}([q \mapsto q(N)(1_A)]_p) = \|\Phi\| \cdot \mu_X^{A,B,C}(p)(N)(1_A) = 0, \tag{44}$$

because $\nu_{\mu(p)}$ dominates the instrument $\mu_X^{A,B,C}(p)$. Hence $K(\Phi) = 0$.

By decomposing into positive and negative real and imaginary parts, the same conclusion holds for arbitrary bounded fields Φ which vanish $\nu_{\mu(p)}$-almost everywhere. Thus K factors through the quotient by $\nu_{\mu(p)}$. This defines the desired normal extension on the ultraweakly dense class of field elements as $\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)}([\Phi]_{\mu(p)}) \coloneqq K(\Phi)$ and hence by normal continuation on all of $L^\infty(\nu_{\mu(p)}) \otimes \mathcal{A}$.

▷ $\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)}$ *is unital.* By lemma 17,

$$\begin{aligned}\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)}(1_X \otimes 1_\mathcal{A}) &= \overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)}([s_{X,1_\mathcal{A}}]_{\mu(p)}) &(45)\\ &= \mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)(X)(1_\mathcal{A}) = 1_\mathcal{C}. &(46)\end{aligned}$$

$\triangleright$ $\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}}(p)$ *is completely positive.* Consider the matrix amplification $\overline{\mu(p)}^{(n)}$ of $\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}}(p)$, which is uniquely defined by

$$\overline{\mu(p)}^{(n)}([\Phi_{jk}]_{\mu(p)}) = [K(\Phi_{jk})] \tag{47}$$
$$= \left[\overline{p}([q \mapsto \overline{q}([\Phi_{jk}]_q)]_p)\right]_{jk} \tag{48}$$
$$= \overline{p}^{(n)}(\left[[q \mapsto \overline{q}([\Phi_{jk}]_q)]_p\right]_{jk}) \tag{49}$$

and define an intermediate map $T(\Phi) = [q \mapsto \overline{q}([\Phi]_q)]_p$, so that $\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}}(p) = \overline{p} \circ T$.

Now, given the identification $M_n(L^\infty(\nu_{\mu(p)}) \mathbin{\bar{\otimes}} \mathcal{A}) \cong L^\infty(\nu_{\mu(p)}) \mathbin{\bar{\otimes}} M_n(\mathcal{A})$, for any positive element of $M_n(L^\infty(\nu_{\mu(p)}) \mathbin{\bar{\otimes}} \mathcal{A})$ we can pick a bounded ultraweaky measurable representative $\Phi : X \to M_n(\mathcal{A})_+$, which we write as $\Phi(x) = [\Phi_{jk}(x)]_{jk}$.

For each $q \in Q^{\mathcal{A},\mathcal{B}}X$, this determines a field $[\Phi]_q \in M_n(L^\infty(\nu_q) \mathbin{\bar{\otimes}} \mathcal{A}) \cong L^\infty(\nu_q) \mathbin{\bar{\otimes}} M_n(\mathcal{A})$ which is also positive. Then we have that

$$T^{(n)}(\Phi) = [q \mapsto \overline{q}^{(n)}([\Phi]_q)]_p \tag{50}$$

Assume that $f \in L^1(\nu_p)$ and $\phi \in \mathcal{B}_*$ are both positive, then

$$\langle f \otimes \phi, T^{(n)}(\Phi)\rangle = \langle f \otimes \phi, [q \mapsto \overline{q}^{(n)}([\Phi]_q)]_p\rangle \tag{51}$$
$$= \int_X f(x)\langle \phi, \overline{q}^{(n)}([\Phi]_q\rangle \ \nu_p(\mathrm{d}x) \tag{52}$$

which is immediately positive. Therefore, $T^{(n)}$ is completely positive, and hence so is $\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}}(p)$.

$\triangleright$ $\overline{\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}}(p)$ *is normal.* Let $[\Phi_\lambda]_{\mu(p)}$ be a bounded increasing net of positive elements in $L^\infty(\nu_p) \mathbin{\bar{\otimes}} \mathcal{A}$ with supremum $[\Phi]_{\mu(p)}$. For each $q \in Q^{\mathcal{A},\mathcal{B}}X$, normality of $\overline{q}$ gives $\overline{q}([\Phi_\lambda]_q) \uparrow \overline{q}([\Phi]_q)$, then normality of $\overline{p}$ gives

$$\overline{p}([q \mapsto \overline{q}([\Phi_\lambda]_q)]_p) \uparrow \overline{p}([q \mapsto \overline{q}([\Phi]_q)]_p) \tag{53}$$

so that $\overline{\mu(p)}$ is normal.

We have therefore shown that the CPU map $\overline{\mu(p)} : L^\infty(m) \mathbin{\bar{\otimes}} \mathcal{A} \to \mathcal{C}$ is the normal extension of $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)$ which therefore has the NEP. □

Lemma 19. *$\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}} : Q^{\mathcal{B},\mathcal{C}}Q^{\mathcal{A},\mathcal{B}}X \to Q^{\mathcal{A},\mathcal{C}}X$ is measurable.*

Proof. Since each $Q^{\mathcal{A},\mathcal{B}}$ is equipped with the evaluation σ-algebra, it suffices to show that $\mathrm{ev}_{E,\phi,a} \circ \mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}$ is measurable for all $E \in \sigma_X$, $\phi \in \mathcal{C}_*$, and $a \in \mathcal{A}$, i.e. that the map

$$p \longmapsto \mathrm{ev}_{E,\phi,a}\left(\int_{Q^{\mathcal{A},\mathcal{B}}(X)} p(\mathrm{d}q) \circ q\right) \tag{54}$$

is measurable. Note that for any $E \in \sigma_X$, $a \in \mathcal{A}$, and $\phi \in \mathcal{B}_*$,

$$\mathrm{ev}_{\phi,1_E \otimes a}(\overline{\mu}) = \langle \phi, \overline{\mu}(1_E \otimes a)\rangle = \langle \phi, \mu(E)(a)\rangle = \mathrm{ev}_{E,\phi,a}(\mu) \tag{55}$$

which by linearity extends to all simple functions $s = \sum_{k=1}^n 1_{E_k} \otimes a_k$ as $\mathrm{ev}_{\phi,s}(\overline{\mu}) = \sum_{k=1}^n \mathrm{ev}_{E_k,\phi,a_k}(\mu)$. Since the pointwise limit of measurable functions is measurable, it follows that the map $Q^{\mathcal{A},\mathcal{B}}X \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{L}^\infty(\nu) \mathbin{\bar{\otimes}} \mathcal{A}, \mathcal{B}) : p \mapsto \overline{p}$ is measurable. Furthermore,

$$\mathrm{ev}_{E,\phi,a}\left(\int_{Q^{\mathcal{A},\mathcal{B}}(X)} p(\mathrm{d}q)\circ q\right) = \left\langle \phi, \int_{Q^{\mathcal{A},\mathcal{B}}(X)} p(\mathrm{d}q)\circ q(E)(a)\right\rangle \tag{56}$$

$$= \langle \phi, \overline{p}([q\mapsto q(E)(a)]_p)\rangle \tag{57}$$

$$= \mathrm{ev}_{\phi,[q\mapsto q(E)(a)]_p}(\overline{p}), \tag{58}$$

and so the mapping of equation (54) is measurable as the composition of measurable maps. □

Lemma 20. *The family $(\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}})_X$ describes a natural transformation $Q^{\mathcal{B},\mathcal{C}}\circ Q^{\mathcal{A},\mathcal{B}} \to Q^{\mathcal{A},\mathcal{C}}$.*

Proof. This is essentially immediate, and entirely analogous to the Giry case (up to tracking the indices $\mathcal{A},\mathcal{B},\mathcal{C}$). □

4.3 The monad laws

Proposition 21. *For any $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in \mathsf{W}^*_{\mathsf{CPSU}}$ and $X\in \mathsf{Measurable}$, the diagram*

$$\begin{tikzcd} Q^{\mathcal{C},\mathcal{D}}Q^{\mathcal{B},\mathcal{C}}Q^{\mathcal{A},\mathcal{B}}X \arrow[r, "\mu^{\mathcal{B},\mathcal{C},\mathcal{D}}_{Q^{\mathcal{A},\mathcal{B}}X}"] \arrow[d, "Q^{\mathcal{C},\mathcal{D}}\mu^{\mathcal{A},\mathcal{B},\mathcal{C}}_X"'] & Q^{\mathcal{B},\mathcal{D}}Q^{\mathcal{A},\mathcal{B}}X \arrow[d, "\mu^{\mathcal{A},\mathcal{B},\mathcal{D}}_X"] \\ Q^{\mathcal{C},\mathcal{D}}Q^{\mathcal{A},\mathcal{C}}X \arrow[r, "\mu^{\mathcal{A},\mathcal{C},\mathcal{D}}_X"] & Q^{\mathcal{A},\mathcal{D}}X \end{tikzcd} \tag{59}$$

commutes.

Proof. Let $p\in Q^{\mathcal{C},\mathcal{D}}Q^{\mathcal{B},\mathcal{C}}Q^{\mathcal{A},\mathcal{B}}X$, then for any measurable $E\subseteq X$ and $a\in\mathcal{A}$,

$$\mu_X^{\mathcal{A},\mathcal{C},\mathcal{D}}(Q^{\mathcal{C},\mathcal{D}}\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p))(E)(a) = \int_{Q^{\mathcal{A},\mathcal{C}}X} Q^{\mathcal{C},\mathcal{D}}\mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(p)(\mathrm{d}q)\circ q(E)(a) \tag{60}$$

$$= \int_{Q^{\mathcal{B},\mathcal{C}}(Q^{\mathcal{A},\mathcal{B}}X)} p(\mathrm{d}q)\circ \mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(q)(E)(a) \tag{61}$$

$$= \overline{p}\left(\left[q\mapsto \mu_X^{\mathcal{A},\mathcal{B},\mathcal{C}}(q)(E)(a)\right]_p\right) \tag{62}$$

$$= \overline{p}\left(\left[q\mapsto \int_{Q^{\mathcal{A},\mathcal{B}}X} q(\mathrm{d}r)\circ r(E)(a)\right]_p\right) \tag{63}$$

$$= \overline{p}\left([q\mapsto \overline{q}([r\mapsto r(E)(a)]_q)]_p\right) \tag{64}$$

$$= \overline{\mu^{\mathcal{B},\mathcal{C},\mathcal{D}}_{Q^{\mathcal{A},\mathcal{B}}X}(p)}([r\mapsto r(E)(a)]_{\mu(p)}) \tag{65}$$

$$= \int_{Q^{\mathcal{A},\mathcal{B}}X} \mu^{\mathcal{B},\mathcal{C},\mathcal{D}}_{Q^{\mathcal{A},\mathcal{B}}X}(p)(\mathrm{d}r)\circ r(E)(a) \tag{66}$$

$$= \mu_X^{\mathcal{A},\mathcal{B},\mathcal{D}}(\mu^{\mathcal{B},\mathcal{C},\mathcal{D}}_{Q^{\mathcal{A},\mathcal{B}}X}(p))(E)(a) \tag{67}$$

as required, where we have applied lemma 18 to obtain equation (65). □

Proposition 22. *For any $\mathcal{A},\mathcal{B}\in \mathsf{W}^*_{\mathsf{CPSU}}$ and $X\in\mathsf{Measurable}$, the diagrams*

$$\begin{tikzcd} Q^{\mathcal{A},\mathcal{B}}X \arrow[r, "Q^{\mathcal{A},\mathcal{B}}\eta^{\mathcal{A}}_X"] \arrow[dr, "1_{Q^{\mathcal{A},\mathcal{B}}X}"'] & Q^{\mathcal{A},\mathcal{B}}Q^{\mathcal{A},\mathcal{A}}X \arrow[d, "\mu^{\mathcal{A},\mathcal{A},\mathcal{B}}_X"] \\ & Q^{\mathcal{A},\mathcal{B}}X \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Q^{\mathcal{A},\mathcal{B}}X \arrow[r, "\eta^{\mathcal{B}}_{Q^{\mathcal{A},\mathcal{B}}X}"] \arrow[dr, "1_{Q^{\mathcal{A},\mathcal{B}}X}"'] & Q^{\mathcal{B},\mathcal{B}}Q^{\mathcal{A},\mathcal{B}}X \arrow[d, "\mu^{\mathcal{A},\mathcal{B},\mathcal{B}}_X"] \\ & Q^{\mathcal{A},\mathcal{B}}X \end{tikzcd} \tag{68}$$

commute.

Proof. ▷ *Left.* Let $p \in Q^{\mathcal{A},\mathcal{B}}X$, then for any measurable $E \subseteq X$ and $a \in \mathcal{A}$,

$$\begin{aligned}
\mu_X^{\mathcal{A},\mathcal{A},\mathcal{B}}(Q^{\mathcal{A},\mathcal{B}}\eta_X^{\mathcal{A}}(p))(E)(a) &= \int_{Q^{\mathcal{A},\mathcal{A}}X} Q^{\mathcal{A},\mathcal{B}}\eta_X^{\mathcal{A}}(p)(\mathrm{d}q) \circ q(E)(a) &(69)\\
&= \int_X p(\mathrm{d}x) \circ \eta_X^{\mathcal{A}}(x)(E)(a) &(70)\\
&= \overline{p}([\eta_X^{\mathcal{A}}(-)(E)(a)]_p) &(71)\\
&= \overline{p}(1_E \otimes a) &(72)\\
&= p(E)(a), &(73)
\end{aligned}$$

so $\mu_X^{\mathcal{A},\mathcal{A},\mathcal{B}}(Q^{\mathcal{A},\mathcal{B}}\eta_X^{\mathcal{A}}(p)) = p$.

▷ *Right.* Let $p \in Q^{\mathcal{A},\mathcal{B}}X$, then for any measurable $E \subseteq X$ and $a \in \mathcal{A}$,

$$\begin{aligned}
\mu_X^{\mathcal{A},\mathcal{B},\mathcal{B}}(\eta_{Q^{\mathcal{A},\mathcal{B}}X}^{\mathcal{B}}(p))(E)(a) &= \mu_X^{\mathcal{A},\mathcal{B},\mathcal{B}}(\delta_p^{\mathcal{B}})(E)(a) &(74)\\
&= \int_{Q^{\mathcal{A},\mathcal{B}}X} \delta_p^{\mathcal{B}}(\mathrm{d}q) \circ q(E)(a) &(75)\\
&= \overline{\delta_p^{\mathcal{B}}}([q \mapsto q(E)(a)]_{\delta_p}) &(76)\\
&= p(E)(a), &(77)
\end{aligned}$$

where we applied lemma 14 in the last step. It follows that $\mu_X^{\mathcal{A},\mathcal{B},\mathcal{B}}(\eta_{Q^{\mathcal{A},\mathcal{B}}X}^{\mathcal{B}}(p)) = p$. □

Putting all of these results together we have that:

Theorem 23. *(Q, μ, η) is a $\mathsf{W}^*_{\mathsf{CPSU}}$-parameterised monad on* $\mathsf{Measurable}$.

In particular, we obtain immediate embeddings of classical probabilistic processes:

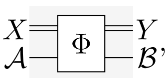
Corollary 24. *$(Q_{\mathbb{C},\mathbb{C}}, \mu^{\mathbb{C},\mathbb{C},\mathbb{C}}, \eta^{\mathbb{C}})$ is the Giry monad.*

5 Quantum Markov kernels

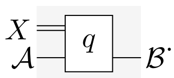
Let $\mathcal{A}, \mathcal{B}$ be W*-algebras, and $(X, \sigma_X), (Y, \sigma_Y)$ be measurable spaces. A *quantum Markov kernel* $(X, \mathcal{A}) \to (Y, \mathcal{B})$ is a mapping $\Phi : \sigma_X \times Y \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B})$ such that

- for each $y \in Y$, the mapping $\sigma_X \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B}) : E \mapsto \Phi(E, y)$ is an NEP quantum instrument; and,
- for each measurable $E \in \sigma_X$, the mapping $Y \to \mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B}) : y \mapsto \Phi(E, y)$ is a measurable map.

Following the quantum computing convention of drawing classical data as doubled wires, we draw quantum Markov kernels $(X, \mathcal{A}) \to (Y, \mathcal{B})$ as diagrams:



, so a quantum instrument is drawn



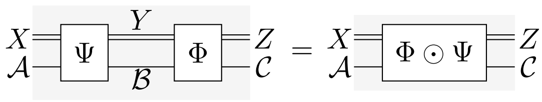
.

Example 25. If X and Y are discrete finite measurable spaces, then a quantum Markov kernel $(X, \mathcal{A}) \to (Y, \mathcal{B})$ can equivalently be seen as a $|Y| \times |X|$ matrix who elements are taken from $\mathsf{W}^*_{\mathsf{CPSU}}(\mathcal{A}, \mathcal{B})$.

We can straightforwardly curry kernels to instrument-valued functions and back:

Proposition 26. *There is a bijection between quantum Markov kernels $(X, \mathcal{A}) \to (Y, \mathcal{B})$ and Kleisli maps $X \to Q^{\mathcal{A},\mathcal{B}}Y$.*

We can therefore obtain the composition of quantum Markov kernels as the composition of the corresponding Kleisli maps:



where explicitly,

$$\Phi \odot \Psi(E, z)(a) = \int_Y \Psi(\mathrm{d}y\,, z) \circ \Phi(E, y)(a), \tag{78}$$

for any $E \in \sigma_X$, $z \in Z$, and $a \in \mathcal{A}$.

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A Monads and parameterised monads

We recall the definitions of monads, Kleisli categories, and parameterised monads. We only overview parameterised monads here, and refer the interested reader directly to [Atk09] of details.

A.1 Monads

Definition 27. *A* monad *on a category* C *consists of an endofunctor* $T : \mathsf{C} \to \mathsf{C}$ *together with natural transformations* $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow T$ *and* $\mu : T^2 \Rightarrow T$*, called the* unit *and* multiplication, *such that the following diagrams commute:*

$$\begin{tikzcd} T^3X \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"'] & TX \end{tikzcd} \tag{79}$$

and

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[dr, "\mathrm{id}_{TX}"'] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "T\eta_X"'] \arrow[dl, "\mathrm{id}_{TX}"] \\ & TX & \end{tikzcd} \tag{80}$$

for every object $X \in \mathsf{C}$.

A.2 Kleisli composition

A monad packages a notion of composition for morphisms of the form $X \to TY$, called *Kleisli morphisms.*

Given morphisms

$$f : X \to TY, \quad \text{and} \quad g : Y \to TZ, \tag{81}$$

their *Kleisli composite* is the morphism

$$g \odot f := X \xrightarrow{f} TY \xrightarrow{Tg} T^2Z \xrightarrow{\mu_Z} TZ. \tag{82}$$

The unit at X, $\eta_X : X \to TX$, acts as the identity Kleisli morphism on X: for every $f : X \to TY$,

$$f \odot \eta_X = f, \quad \text{and} \quad \eta_Y \odot f = f. \tag{83}$$

With this structure, the Kleisli morphisms assemble into a category with the same objects as C, called the *Kleisli category* C_T of the monad T.

A.3 Parameterised monads

Parameterised monads are an indexed version of monads in which processes carry both an input parameter and an output parameter. We state the definition for a parameter category P. In the special case where P is discrete, this reduces to a family indexed by a set.

A *parameterised monad* on C indexed by a category P consists of a functor

$$T : \mathsf{P}^{\mathrm{op}} \times \mathsf{P} \to [\mathsf{C}, \mathsf{C}],$$

written $(p, q) \mapsto T^{p,q}$, together with natural transformations $\eta_p : \mathrm{id}_{\mathsf{C}} \Rightarrow T^{p,p}$ for each $p \in \mathsf{P}$, and $\mu^{p,q,r} : T^{p,q} \circ T^{q,r} \Rightarrow T^{p,r}$ for each composable triple $p, q, r \in \mathsf{P}$, satisfying the associativity and unit axioms below.

For every object $X \in \mathsf{C}$, the multiplication has components $\mu^{p,q,r}_X : T^{p,q}(T^{q,r}X) \to T^{p,r}X$. For every $X \in \mathsf{C}$, the diagram

$$\begin{tikzcd} T^{p,q}T^{q,r}T^{r,s}X \arrow[r, "T^{p,q}(\mu^{q,r,s}_X)"] \arrow[d, "\mu^{p,q,r}_{T^{r,s}X}"'] & T^{p,q}T^{q,s}X \arrow[d, "\mu^{p,q,s}_X"] \\ T^{p,r}T^{r,s}X \arrow[r, "\mu^{p,r,s}_X"'] & T^{p,s}X \end{tikzcd}$$

commutes.

The unit axioms say that, for all $p, q \in \mathsf{P}$, the diagrams

$$
\begin{tikzcd}
T^{p,q}X \arrow[r, "T^{p,q}\eta^q_X"] \arrow[dr, "1_{T^{q,p}X}"'] & T^{p,q}T^{q,q}X \arrow[d, "\mu^{q,q,p}_X"] \\
& Q^{\mathcal{A},\mathcal{B}}X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T^{p,q}X \arrow[r, "\eta^q_{T^{p,q}X}"] \arrow[dr, "1_{T^{p,q}X}"'] & T^{p,p}T^{q,p}X \arrow[d, "\mu^{q,p,p}_X"] \\
& T^{p,q}X
\end{tikzcd}
\tag{84}
$$

commute.

A.4 Parameterised Kleisli composition

Given a parameterised monad T, a parameterised Kleisli morphism from X to Y, changing parameter p to parameter q, is a morphism

$$f : X \to T^{p,q}Y$$

in C.

If

$$f : X \to T^{p,q}Y, \qquad g : Y \to T^{q,r}Z,$$

then their parameterised Kleisli composite is

$$g \odot f := X \xrightarrow{f} T^{p,q}Y \xrightarrow{T^{p,q}g} T^{p,q}T^{q,r}Z \xrightarrow{\mu^{p,q,r}_Z} T^{p,r}Z.$$

This composite is defined only when the output parameter q of f agrees with the input parameter q of g.

Definition 28. *The* parameterised Kleisli category *associated to a parameterised monad T is the category C_T defined as follows:*

- *objects are pairs (p, X), where $p \in \mathsf{P}$ and $X \in \mathsf{C}$;*
- *a morphism $(p, X) \to (q, Y)$ is a morphism $X \to T^{p,q}Y$ in C;*
- *the identity on (p, X) is $\eta^p_X : X \to T^{p,p}X$;*
- *composition is parameterised Kleisli composition.*

The associativity axiom for μ implies associativity of parameterised Kleisli composition. The unit axioms imply that η_p acts as the identity at parameter p. Hence C_T is indeed a category.