A Type Theory for Strictly Unital ∞-Categories

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Abstract

We use type-theoretic techniques to present an algebraic theory of ∞-*categories with strict units*. Starting with a known type-theoretic presentation of fully weak ∞-categories, in which terms denote valid operations, we extend the theory with a non-trivial definitional equality. This forces some operations to coincide strictly in any model, yielding the strict unit behaviour.

We make a detailed investigation of the meta-theoretic properties of this theory. We give a reduction relation that generates definitional equality, and prove that it is confluent and terminating, thus yielding the first decision procedure for equality in a strictly-unital setting. Moreover, we show that our definitional equality relation identifies all terms in a disc context, providing a point comparison with a previously proposed definition of strictly unital ∞-category. We also prove a conservativity result, showing that every operation of the strictly unital theory indeed arises from a valid operation in the fully weak theory. From this, we infer that strict unitality is a property of an ∞-category rather than additional structure.

1 Introduction

Overview One characteristic feature of logical systems based on type theory, and one might say constructive mathematics more generally, is *proof-relevance*. This is to say that these systems manipulate mathematical proofs as first-class entities, in contrast with classical systems which regard proofs as part of meta-mathematics. When applied to logical connectives like implication, conjunction, disjunction, and universal and existential quantification, proof-relevant principles lead to the connection between constructive mathematics and programming languages via the propositions-as-types paradigm.

When applied to *equality*, however, the notion of proof relevance is considerably more subtle. For example, the identity types proposed by Martin-Löf in his eponymous type theory sometimes behave in ways which can be surprising and unintuitive to newcomers. When working in such theories, one quickly arrives at situations where, given two proofs that two elements are equal, one is faced with the task of showing that *these two proofs are themselves equal*, a task which may seem unusual to a classically trained mathematician. Moreover, there are even models where such equality proofs can be unequal [12]. We say in this case that the principle of *uniqueness of identity proofs* fails. Other principles which are often taken for granted classically, such as *function extensionality* and the existence of quotients, may also unavailable.

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But the subtleties of proof-relevant equality are not confined to computer science. Indeed, classical mathematicians have been struggling with them for decades, if from a less foundational point of view. Topologists, for example, quickly realized that it was useful to consider algebraic structures on topological spaces where the axioms held not at the point-set level, but were rather represented as paths in the underlying space. The study of those structures and properties of spaces which depend only on the algebra of paths is called *homotopy theory* [20, 7].

In working with these structures, mathematicians came to discover a major difficulty that arises when handling algebraic objects where equality is given as data, rather than a mere property. When the axioms of an algebraic structure constitute additional structure, then this structure must *itself* be subject to "higher" axioms, colloquially known as *coherence data*. In a proof-relevant setting, this coherence data will itself generate further coherence data, in a tower of increasing complexity, yielding a combinatorial explosion. This is difficult to handle, and it is a non-trivial problem to even *describe* well-behaved algebraic theories in proof-relevant settings.

This difficulty is perhaps best exemplified by the search for a well-behaved theory of higher dimensional category theory [14]. By now this is an active area of research, with many known techniques and proposed definitions. Moreover, the theory has applications in areas as diverse as topology, manifold theory, representation theory, algebra and even fundamental physics [2].

The realization that the techniques and ideas which apply in the study of homotopy theory and higher category theory apply equally well to the proof-relevant versions of equality found in computer science is the central intuition of *Homotopy Type Theory* [19]. And indeed, importing ideas from homotopy theory has proven fruitful for understanding the proof-relevant equality of type theory, leading to new ideas and resolutions of some of the problems described above [9].

But conversely, ideas from logic and computer science may also be applied to the description and manipulation of higher dimensional structures. While classically such structures are often developed from *combinatorial* descriptions of their underlying data (for example, using *simplices* [15]), the connections with logic suggest an alternative *syntactic* approach, closer to that of universal algebra. The problem of presenting higher-dimensional categories, for example, has recently been given a syntactic description in the type theory Catt [10]. This yields an explicit machine-checkable syntax for higher-categorical operations, which applies in principle in arbitrary dimensions.

But beyond the initial problem of describing proof-relevant structures lies a deeper one: actually using and the resulting definitions in practice. Fans of computer formalization can surely attest to the fact that proof-relevant theories can sometimes be extremely verbose. This can be especially frustrating, because it is often intuitively clear that much of the data ought to be *redundant*, and hence recoverable in some automatic way. This is not lost on mathematicians, and many theorems in the literature [11, 16] show that certain higher-categorical structures can be "strictified". The search for presentations of higher-categorical structures in which certain operations hold strictly, but which retain the same expressive power, goes by the name *semi-strict* higher category theory.

Our Contribution In this article, we apply type-theoretic ideas to reduce the complexity of the tower of coherence data in the definition of higher-dimensional category. Our approach is to equip the type theory Catt [10] with a non-trivial definitional equality, written "$=$", which strictifies the part of the theory that handles composition with units, and its associated higher coherence data. We call this new theory $\mathsf{Catt}_{\mathsf{su}}$, standing for

"Catt with strict units".

Since the terms of Catt describe higher-categorical operations, this definitional equality yields a new theory where families of operations coincide. For example, in a fully weak higher category, the equation

$$f \circ \mathrm{id}_x = f$$

would need to be witnessed by explicit coherence data. But in $\mathsf{Catt}_{\mathsf{su}}$ these terms are definitionally equal, and so this unit law holds on-the-nose, and the associated tower of higher-dimensional coherence data trivialises.

We give a detailed analysis of definitional equality in $\mathsf{Catt}_{\mathsf{su}}$, via a reduction strategy that reduces the complexity of terms. We show that this terminates after finitely many steps, and hence yields for every term t a unique normal form $N(t)$. We then show that this generates definitional equality, in the sense that $t = t'$ if and only if $N(t) \equiv N(t')$, where "$\equiv$" represents syntactic equality up to α-equivalence. The existence of distinct normal forms also shows our theory has non-trivial models.

Definitional equality is hence decidable, and the type theory can be implemented. Our OCaml implementation is made available online at the following address:

`http://github.com/ericfinster/catt.io/tree/v0.1`

Our results yield the first type-theoretic definition of strictly unital ∞-categories. Previous work has suggested that in such a theory, any pair of valid terms over a disc context should be equal. We prove this property directly for our definition, giving a useful "sanity check".

Since our theory $\mathsf{Catt}_{\mathsf{su}}$ is obtained by adding a definitional equality relation to Catt, there is a trivial mapping $K : \mathsf{tm}(\mathsf{Catt}) \to \mathsf{tm}(\mathsf{Catt}_{\mathsf{su}})$. That is, every Catt term is automatically a $\mathsf{Catt}_{\mathsf{su}}$ term. This raises the important question: for every $\mathsf{Catt}_{\mathsf{su}}$ term t, can we find a Catt term t' such that $t = K(t')$? We show that this holds for terms over pasting diagrams, thus verifying a form of *conservativity*: every operation in the semi-strict theory corresponds to one in the original weak theory in this case. We apply this result to show that for a weak ∞-category, having strict units is a *property*, rather than extra structure.

Finally, we use our implementation to construct two substantial examples of terms in our theory, inspired by important algebraic structures in homotopy theory.

– The *Eckmann-Hilton move* or *braiding*, is a 3-dimensional term that has been previously constructed in Catt [10]. We apply our reduction relation to find its $\mathsf{Catt}_{\mathsf{su}}$ normal form, obtaining a proof object which is over 50 times smaller than the original.

– The second example, called the *syllepsis*, is a 5-dimensional algebraic object that had not been explicitly constructed in a pure language of path types prior to this work being advertised.[1] We give a formalization of the syllepsis in $\mathsf{Catt}_{\mathsf{su}}$. Due to its complexity, we have not been able to construct the syllepsis in Catt.

This demonstrates our main research goal: to simplify the syntax of higher category theory so that proofs become smaller, and proof construction becomes easier.

1.1 Related work

Homotopy Type Theory While our work is not directly concerned with Homotopy Type Theory (HoTT) [19], it is nonetheless heavily inspired by developments that these

[1] One could say that it had been indirectly constructed in homotopy type theory, through a calculation of $\pi_4(S^3)$ [8]. Since our work has been completed and advertised, pure path models of the syllepsis have now been constructed by some members of the homotopy type theory community [18].

ideas have provoked in the type theory community. Indeed, the definition of ∞-category presented in [10], and developed in the present work, was based on a similar definition of ∞-groupoid [8], which in turn can be seen as a distillation of exactly that part of Martin-Löf's identity elimination principle which causes types to behave as higher-dimensional groupoids.

We emphasize that our theory is relevant for directed higher categories, while HoTT is currently only able to reason about higher groupoids. At the same time, HoTT is a rich logical system with sums, products and other higher type formers, while our theory is more restricted, providing only the type formers for building categorical operations.

Higher Category Theory Our work extends the type theory Catt of Finster and Mimram, presented at LICS 2017 [10]. Its models yield an algebraic notion of weak ∞-category which agrees with a definition due to Maltsiniotis [17], which itself is a close cousin of Batanin's [4] and Leinster's [14] definitions; see [1] for a direct comparison.

A previous definition of strictly unital weak ∞-category has been given by Batanin, Cisinski and Weber [5], using the mathematical language of operads. That theory has two main axiom classes, *disk reduction* and *unit compatibility*, and we can compare these to our three main generators for definitional equality, *prune*, *disk* and *endo*. While the precise relationship between the theories is not completely clear, we make the following observations.

– Their *unit compatibility* axiom is close to our *prune* generator, a key point of similarity between the theories.

– Their *disk reduction* axiom simply requires that terms trivialize over disks, while for us the corresponding statement is a theorem (see Theorem 45.) That is, an axiom of BCW becomes an *emergent property* in our setting.

– Our *endo* axiom appears to have no parallel in BCW's theory, and we are able to exhibit a pair of terms which are definitionally equal here, but we believe would *not* be identified in their theory (see page 14 here.) In this sense, we claim that our theory is likely to be stricter.

In general, our approach emphasizes an explicit syntax for the operations of our theory which we are able to manipulate on a computer, while the BCW approach might be described as more "semantic", characterizing the theory by universal properties, but without an explicit description of the resulting operations.

2 The Type Theories Catt and $\mathsf{Catt}_{\mathsf{su}}$

It will be convenient to construct our theory in three layers. We begin with the raw syntax and basic rules for contexts, types and substitutions, leaving out the term forming rules. Most of this material is standard, and we make sure to point out any idiosyncracies of notation as we proceed. After introducing the notion of *pasting context* we then present the term forming structure of Catt. Finally, we introduce some combinatorial material necessary to describe our equality relation on terms, culminating in the definition of the theory $\mathsf{Catt}_{\mathsf{su}}$.

2.1 The Base Theory Catt

Here we present the type theory Catt defined by Finster and Mimram [10].

Raw Syntax We fix an infinite set V of variables, and use lowercase Roman $(x, y, \dots)$ and Greek $(\alpha, \beta, \dots)$ letters to refer to its elements. The raw syntax of Catt consists of four syntactic classes: *contexts*, *types*, *terms* and *substitutions* (denoted Ctx, Type, Term and Sub, respectively). These classes are defined by the rules in Figure 1. The arrow constructor $s \to_A t$ for types represents the type of directed paths from s to t where s and t have type A. This is similar to the equality type $s =_A t$ in Martin-Löf Type Theory which can be thought of as the type of undirected paths between s and t. Observe that both contexts and substitutions appear in the raw syntax of terms.

We write $\equiv$ for syntactic equality of the various syntactic classes up to α-equivalence.

Free Variables The *free variables* of elements of each syntactic class are defined by induction on the structure as follows, where $x \in V$:

$$\begin{aligned} \mathsf{FV}(\emptyset) &= \emptyset & \mathsf{FV}(\Gamma, x : A) &= \mathsf{FV}(\Gamma) \cup \{x\} \\ \mathsf{FV}(\star) &= \emptyset & \mathsf{FV}(s \to_A t) &= \mathsf{FV}(A) \cup \mathsf{FV}(s) \cup \mathsf{FV}(t) \\ \mathsf{FV}(x) &= \{x\} & \mathsf{FV}(\mathsf{coh}\,(\Gamma : A)[\sigma]) &= \mathsf{FV}(\sigma) \\ \mathsf{FV}(\langle\rangle) &= \emptyset & \mathsf{FV}(\langle \sigma, x \mapsto t \rangle) &= \mathsf{FV}(\sigma) \cup \mathsf{FV}(t) \end{aligned}$$

Dimension We define the *dimension* of a type by induction:

$$\mathsf{dim}\,\star = -1 \qquad\qquad \mathsf{dim}\,(s \to_A t) = \mathsf{dim}\,A + 1$$

We extend this notion to contexts by asserting that the dimension of a context is one more than the maximum of the dimension of the types occurring in that context.

$$\mathsf{dim}\,\emptyset = -1 \qquad\qquad \mathsf{dim}\,(\Gamma, x : A) = \max(\mathsf{dim}\,\Gamma, \mathsf{dim}\,A + 1)$$

Term Substitutions and Compositions Although the terms of Catt are always in normal form, we will need to perform actual substitutions on terms during type-checking. We therefore also define a semantic form of substitution which calculates by induction on the structure of terms. We denote this operations by $[\![-]\!]$ in order to distinguish if from the

$$\frac{}{\emptyset : \mathsf{Ctx}} \qquad \frac{\Gamma : \mathsf{Ctx} \qquad A : \mathsf{Type}}{\Gamma, A : \mathsf{Ctx}}$$

$$\frac{}{\star : \mathsf{Type}} \qquad \frac{A : \mathsf{Type} \qquad s : \mathsf{Term} \qquad t : \mathsf{Term}}{s \to_A t : \mathsf{Type}}$$

$$\frac{v : V}{v : \mathsf{Term}} \qquad \frac{\Gamma : \mathsf{Ctx} \qquad A : \mathsf{Type} \qquad \sigma : \mathsf{Sub}}{\mathsf{coh}\,(\Gamma : A)[\sigma] : \mathsf{Term}}$$

$$\frac{}{\langle\rangle : \mathsf{Sub}} \qquad \frac{\sigma : \mathsf{Sub} \qquad t : \mathsf{Term}}{\langle \sigma, t \rangle : \mathsf{Sub}}$$

Figure 1: Raw syntax

$[-]$ appearing in coherence terms, which is part of the syntactic structure. This operation is defined as follows:

$$\begin{aligned}
\star[\![\sigma]\!] &= \star \\
(s \to_A t)[\![\sigma]\!] &= s[\![\sigma]\!] \to_{A[\sigma]} t[\![\sigma]\!] \\
\mathsf{coh}\,(\Gamma : s \to t)[\tau][\![\sigma]\!] &= \mathsf{coh}\,(\Gamma : s \to t)[\tau \circ \sigma] \\
x[\![\sigma]\!] &= t \qquad \text{if } x \mapsto t \in \sigma
\end{aligned}$$

Composition of substitutions, written "$\circ$", is defined mutually recursively as follows:

$$\langle\rangle \circ \sigma = \langle\rangle \qquad\qquad \langle \tau, t\rangle \circ \sigma = \langle \tau \circ \sigma, t[\![\sigma]\!]\rangle$$

For every context Γ we also have an identity substitution id_Γ, mapping each variable to itself. By simple induction we get the following proposition.

Proposition 1. *Contexts and substitutions form a category; that is, the following syntactic equations hold, for context Γ, type A and term s of Γ, and composable substitutions μ, σ, τ:*

$$\begin{aligned}
A[\![\sigma \circ \tau]\!] &\equiv A[\![\sigma]\!][\![\tau]\!] & A[\![\mathrm{id}_\Gamma]\!] &\equiv A \\
s[\![\sigma \circ \tau]\!] &\equiv s[\![\sigma]\!][\![\tau]\!] & s[\![\mathrm{id}_\Gamma]\!] &\equiv s \\
\mu \circ (\sigma \circ \tau) &\equiv (\mu \circ \sigma) \circ \tau & \mathrm{id}_\Delta \circ \mu &\equiv \mu \equiv \mu \circ \mathrm{id}_\Gamma
\end{aligned}$$

Proof. This is a simple mutual induction over the syntax. □

The basic typing judgements for contexts, types and substitutions are given in Figure 2. The rules are standard for a dependent type theory. Note that our types consist of just a single base type denoted $\star$, and a formation rule analogous to the formation rule for identity types in Martin Löf Type Theory. The fact that this rule captures faithfully the notion of globular set is at the heart of the connection between type theory and higher category theory, and is the basis of this syntactic description of ∞-categories.

Support Given a term t and context Γ, we can define the *support* of t in Γ to be the downwards closure of $FV(t)$ in Γ, where a set of variables S is downwards closed if for all $x \in S$, $x : A \in \Gamma$, and $y \in \mathsf{FV}(A)$ we have $y \in S$. If the context is obvious we simply write $\mathsf{supp}(t)$ for the support of t.

Pasting Contexts The terms of Catt are derived from isolating a distinguished subset of contexts which we call *pasting contexts*. A set of rules for exhibiting evidence that a given context is a pasting context was the key innovation of [10]. These rules are presented in Figure 3.

$$\frac{}{\emptyset \vdash} \qquad\qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A : \mathsf{Type}}{\Gamma, x : A \vdash}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \mathsf{Type}} \qquad\qquad \frac{\Gamma \vdash A : \mathsf{Type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a \to_A b : \mathsf{Type}}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \langle\rangle : \emptyset} \qquad\qquad \frac{\Gamma \vdash \sigma : \Delta \qquad \Delta \vdash A : \mathsf{Type} \qquad \Gamma \vdash t : A[\sigma]}{\Gamma \vdash \langle \sigma, x \mapsto t\rangle : \Delta, x : A}$$

Figure 2: Basic typing rules

Boundary Variables For each pasting context $\Gamma \vdash_p$, we will define two distinguished subsets of the variables, denoted $\partial^-(\Gamma)$ and $\partial^+(\Gamma)$. First, for a variable $x : A \in \Gamma$, define its dimension to be $\mathsf{dim}\, A + 1$. Furthermore, let us say that a variable is *target-free* if it does *not* occur as the target of any other variables in Γ. Similarly, we have the notion of *source-free*. We now define:

$$\begin{aligned}
\partial^-(\Gamma) := \{x \in \Gamma \mid\, & \mathsf{dim}\, x < \mathsf{dim}\, \Gamma - 1, \\
& \textit{or, } \mathsf{dim}\, x = \mathsf{dim}\, \Gamma - 1 \text{ and } x \text{ is target-free}\} \\
\partial^+(\Gamma) := \{x \in \Gamma \mid\, & \mathsf{dim}\, x < \mathsf{dim}\, \Gamma - 1, \\
& \textit{or, } \mathsf{dim}\, x = \mathsf{dim}\, \Gamma - 1 \text{ and } x \text{ is source-free}\}
\end{aligned}$$

Terms With these notions in place, the typing rules for terms of Catt are shown in Figure 4. Note that when writing the substitution in a coherence term, we typically omit the angled brackets, writing $\mathsf{coh}\,(\Gamma : s \to_A t)[a, b, c, \dots]$ instead of $\mathsf{coh}\,(\Gamma : s \to_A t)[\langle a, b, c, \dots \rangle]$.

Examples We record here some basic examples of well-typed terms.

$$\begin{aligned}
\mathsf{comp}_1 &:= \mathsf{coh}\,((x : \star)(y : \star)(f : x \to_\star y) : x \to_\star y)[x, y, f] \\
\mathsf{comp}_2 &:= \mathsf{coh}\,((x : \star)(y : \star)(f : x \to_\star y)(z : \star)(g : y \to_\star z) \\
&\qquad : x \to_\star z)[x, y, f, z, g] \\
\mathsf{comp}_3 &:= \mathsf{coh}\,((x : \star)(y : \star)(f : x \to_\star y)(z : \star)(g : y \to_\star z) \\
&\qquad (w : \star)(h : z \to_\star w) : x \to_\star w)[x, y, f, z, g, w, h] \\
\mathbb{1}_0 &:= \mathsf{coh}\,((x : \star) : x \to_\star x)[x] \\
\mathsf{unit\text{-}r} &:= \mathsf{coh}\,((x : \star)(y : \star)(f : x \to_\star y) \\
&\qquad : \mathsf{comp}_2[\![x, y, f, y, \mathbb{1}_0[\![y]\!]]\!] \to_{x \to_\star y} f)[x, y, f]
\end{aligned}$$

$$\frac{}{x : \star \vdash_p x : \star}\ \star \qquad \frac{\Gamma \vdash_p x : \star}{\Gamma \vdash_p}\ \checkmark$$

$$\frac{\Gamma \vdash_p x : A}{\Gamma, y : A, f : x \to_A y \vdash_p f : x \to_A y}\ \Uparrow \qquad \frac{\Gamma \vdash_p f : x \to_A y}{\Gamma \vdash_p y : A}\ \Downarrow$$

Figure 3: Pasting contexts

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}$$

$$\frac{\Gamma \vdash_p \qquad \Gamma \vdash s \to_A t \qquad \Delta \vdash \sigma : \Gamma \qquad \mathsf{supp}(s) = \partial^-(\Gamma) \qquad \mathsf{supp}(t) = \partial^+(\Gamma)}{\Delta \vdash \mathsf{coh}\,(\Gamma : s \to_A t)[\sigma] : s[\![\sigma]\!] \to_{A[\sigma]} t[\![\sigma]\!]}$$

$$\frac{\Gamma \vdash_p \qquad \Gamma \vdash s \to_A t \qquad \Delta \vdash \sigma : \Gamma \qquad \mathsf{supp}(s) = \mathsf{supp}(t) = \mathsf{FV}(\Gamma)}{\Delta \vdash \mathsf{coh}\,(\Gamma : s \to_A t)[\sigma] : s[\![\sigma]\!] \to_{A[\sigma]} t[\![\sigma]\!]}$$

Figure 4: Term constructors

$$\begin{aligned}\mathsf{comp}_{2,0} := \mathsf{coh}\,(&(x:\star)(y:\star)(f:x\to_\star y)(z:\star)(g:y\to_\star z)\\&(h:y\to_\star z)(\alpha:g\to_{y\to_\star z} h)(w:\star)(k:z\to_\star w)\\&:\mathsf{comp}_3[\![x,y,f,z,g,w,k]\!]\to_{(x\to_\star w)}\\&\ \ \mathsf{comp}_3[\![x,y,f,z,h,w,k]\!])[x,y,f,z,g,h,\alpha,w,k]\end{aligned}$$

In the theory of ∞-categories, these correspond to *operations* from low-dimensional higher category theory, such as the theory of bicategories. In that more familiar language, given objects x, y, z, w and 1-morphisms $f : x \to y$, $g : y \to z$, $h : z \to w$, we have the following, where we write $\cdot$ for forward composition of 1-morphisms:

- $\mathsf{comp}_1[\![x, y, f]\!]$ gives the unary composite (f);
- $\mathsf{comp}_2[\![x, y, f, z, g]\!]$ gives the binary composite $f \cdot g$;
- $\mathsf{comp}_3[\![x, y, f, z, g, w, h]\!]$ gives the unbiased ternary composite $f \cdot g \cdot h$;
- $\mathsf{comp}_2[\![x, z, \mathsf{comp}_2[\![x, y, f, z, g]\!], w, h]\!]$ gives the iterated binary composite $(f \cdot g) \cdot h$;
- $\mathbb{1}_0[\![x]\!]$ corresponds to the 1-morphism id_x;
- $\mathsf{unit\text{-}r}[\![f]\!]$ gives to the unit 2-morphism $f \cdot \mathrm{id}_y \Rightarrow f$.

The unary composite (f) and the unbiased ternary composite $f \cdot g \cdot h$ are not directly defined in the traditional notion of bicategory. We could similarly write down operations for other familiar operations in the theory of weak ∞-categories, such as associators, interchangers, and so on, in principle in all dimensions. It is in this sense that Catt gives a formal language for weak ∞-categories.

This last example $\mathsf{comp}_{2,0}$ is part of a family of coherences $\mathsf{comp}_{d,k}$ for $k < d$, which will play a role in Section 6. While we will not give a formal definition, these compositions can be described intuitively as follows: they consist of the "unbiased" composite of a d-dimensional disc D with two $(k+1)$-dimensional discs S and T glued to the k-dimensional source and target of D, respectively. In traditional notation we might write this as $S \circ_k D \circ_k T$.

Identity Terms Generalizing the 0-dimensional case above, we can define an identity on cells of arbitrary dimension. To do so, we assume that the set V of variables contains elements d_i and d'_i for $i \in \mathbb{N}$. Now define the *k-disc context* and the *$(k-1)$-sphere type* by mutual induction on k as follows:

$$\mathrm{D}^0 := \emptyset, (d_0 : \star) \qquad\qquad S^{-1} := \star$$
$$\mathrm{D}^{k+1} := \mathrm{D}^k, (d'_k : S^{k-1}), (d_{k+1} : S^k) \qquad\qquad S^k := d_k \to_{S^{k-1}} d'_k$$

Finally, for $k \in \mathbb{N}$, we define the *identity on the k-disc*, written $\mathbb{1}_k$ as follows, a valid term in the context D^k:

$$\mathbb{1}_k := \mathsf{coh}\,(\mathrm{D}^k : d_k \to_{S^{k-1}} d_k)[\mathrm{id}_{D^k}]$$

Substitutions From a Disc A substitution out of a k-disc context contains the same data as a type and a term. Given a type A and term t, we define a substitution $\{A, t\}$ as follows:

$$\{\star, x\} := \langle x\rangle \qquad\qquad \{u \to_A v, x\} := \langle \{A, u\}, v, x\rangle$$

Moreover, given a substitution σ from a disc D^k, we have $\sigma \equiv \{S^{k-1}[\![\sigma]\!], d_k[\![\sigma]\!]\}$. We also have the following lemma describing the interaction between substitutions from a disc and composition.

Lemma 2. *Substitutions from a disc compose in the following way:*

$$\{A[\![\sigma]\!], t[\![\sigma]\!]\} \equiv \{A, t\} \circ \sigma$$

Proof. If $A \equiv \star$ then the lemma reduces to $\langle t \rangle \equiv \langle t \rangle$. If $A \equiv u \to_{A'} v$, then we reason by induction on subtypes as follows:

$$\begin{aligned}
\{A[\![\sigma]\!], t[\![\sigma]\!]\} &\equiv \{u[\![\sigma]\!] \to_{A'[\sigma]} v[\![\sigma]\!], t[\![\sigma]\!]\} \\
&\equiv \langle \{A'[\![\sigma]\!], u[\![\sigma]\!]\}, v[\![\sigma]\!], t[\![\sigma]\!] \rangle \\
&\equiv \langle \{A', u\}[\![\sigma]\!], v[\![\sigma]\!], t[\![\sigma]\!] \rangle \\
&\equiv \langle \{A', u\}, v, t \rangle \circ \sigma \\
&\equiv \{u \to_{A'} v, t\} \circ \sigma \\
&\equiv \{A, t\} \circ \sigma
\end{aligned}$$

This completes the proof. $\square$

Inferred Types If a term t is well-scoped in some context Γ (i.e. all variables it contains are either bound or in Γ), we can derive a canonical type for it $\mathsf{ty}(t)$, which we call the *inferred type*. For a variable x, by assumption there is $(x, A) \in \Gamma$ for some type A, and we define $\mathsf{ty}(x) := A$. For a coherence term, we define the inferred type as follows:

$$\mathsf{ty}(\mathsf{coh}\,(\Gamma : U)[\sigma]) := U[\![\sigma]\!]$$

Furthermore, when the inferred type of a term u is of the form $s \to_A t$, we define the *inferred source*, $\mathsf{src}(u)$, to be s and the *inferred target* $\mathsf{tgt}(u)$ to be t. We note that a term of the form $\mathsf{coh}\,(\Gamma : \star)[\ldots]$ is never valid, so $\mathsf{src}, \mathsf{tgt}$ are defined for every valid coherence term.

We write $\mathsf{src}^k, \mathsf{tgt}^k$ for the iterated k-fold inferred source or target, and src_k, tgt_k for the k-dimensional inferred source or target; so for a coherence term t of dimension n, we have $\mathsf{src}_k(t) := \mathsf{src}^{n-k}(t)$, and $\mathsf{tgt}_k(t) := \mathsf{tgt}^{n-k}(t)$.

Using inferred types we can extend the notion of dimension to terms, by defining $\dim t := \dim \mathsf{ty}(t) + 1$.

2.2 The type theory $\mathsf{Catt}_{\mathsf{su}}$

As described in the introduction, the type theory Catt of the previous section contains no non-trivial definitional equalities: while calculation happens during type-checking, all terms themselves are in normal form. In this section, we introduce our equality relation. Its definition will require some combinatorial preparation which we turn to now.

Dyck Words Observe that each of the rules for pasting contexts in Figure 3 has at most one hypothesis, and consequently, derivations made with these rules are necessarily linear. In fact, complete derivations of the fact that a context is a pasting context can be identified with *Dyck words* [13]. Strictly speaking, a pasting context is defined as a pair of a syntactic

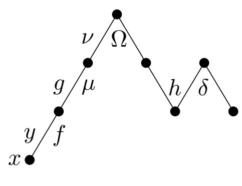


Figure 5: An element of$\mathsf{Dyck}\,1 \Downarrow (\Uparrow (\Downarrow (\Downarrow (\Uparrow (\Uparrow (\Uparrow (\star x)\, y\, f)\, g\, \mu)\, \nu\, \Omega)))\, h\, \delta)$

context and a derivation of the fact that it is well-formed, but this representation contains redundancy and can be awkward to manipulate in practice. For presentation, it is more convenient to work with *Dyck words*, which are a somewhat simplified representation of pasting diagrams.

Dyck words may be pictured as a list of up and down moves, with each up move labeled by a pair of variable names. Concretely, the set $\mathsf{Dyck}\, n$ of *Dyck words of excess* n is defined by the rules given in Figure 6. The parameter n records the difference between the number of up and down moves. The definition ensures that the excess is always non-negative, so we always have at least as many up moves as down moves, leading to the "mountain" diagram of Figure 5.

The rules for Dyck words mirror exactly the derivation rules for pasting contexts of Figure 3. (The additional ✓ rule for pasting contexts forces a complete derivation $\Gamma \vdash_p$ to be of excess 0). As an example, consider this context:

$$(x : *)(y : *)(f : x \to_\star y)(z : *)(g : y \to_\star z)$$

It is proven to be a pasting context via this derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(x:\star) \vdash_p (x:\star)}\,\star}{(x:\star)(y:\star)(f : x \to_\star y) \vdash_p (f : x \to_\star y)}\,\Uparrow}{(x:\star)(y:\star)(f : x \to_\star y) \vdash_p (y:\star)}\,\Downarrow}{(x:\star)(y:\star)(f : x \to_\star y)(x:\star)(g : y \to_\star z) \vdash_p (g : y \to_\star z)}\,\Uparrow}{(x:\star)(y:\star)(f : x \to_\star y)(x:\star)(g : y \to_\star z) \vdash_p (z:\star)}\,\Downarrow$$

Its Dyck word representation is the following:

$$\Downarrow\ (\Uparrow\ (\Downarrow\ (\Uparrow\ (\star\, x)\; y\; f))\; z\; g)) : \mathsf{Dyck}\, 0$$

The only difference between these two representations is that the pasting context, together with its derivation, remembers all the typing information of the all variables, while the Dyck word representation remembers just the variable names. Since the types can be recovered from the structure of the Dyck word itself, this represents no loss of information. To fix notation, if $\Gamma \vdash_p (x : A)$ is a pasting context, we write $\lfloor \Gamma \rfloor : \mathsf{Dyck}\,(\mathsf{dim}\, A)$ for the corresponding Dyck word. Conversely, for a Dyck word $d : \mathsf{Dyck}\, n$, we write $\lceil d \rceil$, $\mathsf{ty}(d)$, and $\mathsf{tm}(d)$ for the corresponding context, type and term such that $\lceil d \rceil \vdash_p \mathsf{tm}(d) : \mathsf{ty}(d)$.

$$\dfrac{x \in V}{\star\, x : \mathsf{Dyck}\, 0} \qquad \dfrac{n : \mathbb{N} \qquad d : \mathsf{Dyck}\, n \qquad y, f : V}{\Uparrow\ d\, y\, f : \mathsf{Dyck}\,(S\, n)} \qquad \dfrac{n : \mathbb{N} \qquad d : \mathsf{Dyck}\,(S\, n)}{\Downarrow\ d : \mathsf{Dyck}\, n}$$

Figure 6: Dyck words

Peaks A special role will be played by the positions in a Dyck word where we change direction from moving up to moving down. We call these the *peaks*. It is easy to give an induction characterization of peaks, and we do so in Figure 7. A nice advantage of this representation is that we can write programs on pasting contexts using pattern matching style (for which we use an Agda-style syntax).

If $\Gamma \vdash_p$ is a pasting context, we say that a variable which occurs as the label of a peak in the Dyck word representation of Γ is *locally maximal*. Intuitively speaking, such variables represent those cells which are of highest dimension "in their neighborhood." We write $\mathsf{LM}(\Gamma)$ for the set of locally maximal variables and for $\alpha \in \mathsf{LM}(\Gamma)$ we write p_α for the corresponding peak. Observe that if $\alpha \in \mathsf{LM}(\Gamma)$, we may speak of the *dimension* $\mathsf{dim}\,\alpha$ since α is assigned a type by Γ. Moreover, since all locally maximal variables are positive dimensional, the inferred type of any locally maximal variable $\alpha \in \mathsf{LM}(\Gamma)$ must be an arrow type $\alpha : s \to_A t$ for some type A. Therefore the src and tgt of a locally maximal are always well defined. We note that any pasting context Γ contains no coherence terms, and so $\mathsf{src}(\alpha)$ and $\mathsf{tgt}(\alpha)$ are always variables.

We can use the definition of peaks to manipulate our Dyck word in various ways. We define three operations in Figure 8 which will be useful in what follows.

The first operation is *excising* a peak of a Dyck word. Given a pasting context $\Gamma \vdash_p$ and a locally maximal variable $\alpha : \mathsf{LM}(\Gamma)$, we write $\Gamma /\!/ \alpha$ for the pasting context obtained by excising the peak where α occurs in the Dyck word representation of Γ:

$$\Gamma /\!/ \alpha := \lceil \mathsf{excise}\, p_\alpha \rceil$$

This operation removes α and $\mathsf{tgt}(\alpha)$ from the original context. A pictorial representation is given in Figure 9.

The second operation, *project*, is used to create a natural substitution $\Gamma /\!/ \alpha \vdash \pi_\alpha : \Gamma$, given by:

$$\pi_\alpha := \mathsf{project}\, p_\alpha$$

If $\mathsf{ty}(\alpha) \equiv \mathsf{src}(\alpha) \to_A \mathsf{tgt}(\alpha)$ then this sends the variable α to $\mathbb{1}_{\mathsf{dim}\,A}[\![\{A, \mathsf{src}(\alpha)\}]\!]$, the variable $\mathsf{tgt}(\alpha)$ to $\mathsf{src}(\alpha)$, and every other variable to itself.

Lastly we also define an operation, *remove*, that removes two terms from a given substitution. If Δ is another context, and we are given a substitution $\Delta \vdash \sigma : \Gamma$, then we have a substitution $\Delta \vdash \sigma /\!/ \alpha : \Gamma /\!/ \alpha$ obtained by the remove function:

$$\sigma /\!/ \alpha := \mathsf{remove}\, p_\alpha\, \sigma$$

which removes the terms in σ corresponding to α and $\mathsf{tgt}(\alpha)$.

We will show that these are actually well typed constructions later in Lemma 11.

Equality in $\mathsf{Catt_{su}}$ With these definitions in place, we can now define our equality relation on terms in Figure 10. Our equality is indexed by the context which we are working with, and we will write statements like $\Gamma \vdash s = t$ to mean terms s and t are equal in context Γ. Where the context is clear we may drop it and simply write $s = t$. There are three generating equalities, which we summarize as follows.

- *Pruning* scans the locally maximal arguments of a substitution looking for identity terms. When such an argument appears, it may be removed, while at the same time removing the corresponding Dyck peak from the pasting diagram defining the coherence.

$$\frac{n : \mathbb{N} \quad d : \mathsf{Dyck}\, n \quad y, f : V}{\Updownarrow_{\mathsf{pk}}\ d : \mathsf{Peak}\,(\Downarrow (\Uparrow\ d\, y\, f))} \qquad \frac{n : \mathbb{N} \quad d : \mathsf{Dyck}\, n \quad y, f : V \quad p : \mathsf{Peak}\, d}{\Uparrow_{\mathsf{pk}}\ d\, y\, f\, p : \mathsf{Peak}\,(\Uparrow\ d\, y\, f)}$$

$$\frac{n : \mathbb{N} \quad d : \mathsf{Dyck}\,(S\, n) \quad p : \mathsf{Peak}\, d}{\Downarrow_{\mathsf{pk}}\ d\, p : \mathsf{Peak}\,(\Downarrow\ d)}$$

Figure 7: Peaks

$$
\begin{aligned}
&\mathsf{excise} : \{n : \mathbb{N}\}\{d : \mathsf{Dyck}\,n\}(p : \mathsf{Peak}\,d) \to \mathsf{Dyck}\,n\\
&\mathsf{excise}\,(\Updownarrow_{pk}\ d\,y\,f) = d\\
&\mathsf{excise}\,(\Uparrow_{pk}\ d\,y\,f\,p) = \Uparrow (\mathsf{excise}\,p)\,y\,f\\
&\mathsf{excise}\,(\Downarrow_{pk}\ d\,p) = \Downarrow (\mathsf{excise}\,p)
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{project} : \{n : \mathbb{N}\}\{d : \mathsf{Dyck}\,n\}(p : \mathsf{Peak}\,d) \to \mathsf{Sub}\\
&\mathsf{project}\,(\Updownarrow_{pk}\ d\,y\,f) = \langle \mathrm{id}_{\lceil d \rceil}, \mathsf{tm}(d), \mathbb{1}(\mathsf{ty}(d), \mathsf{tm}(d))\rangle\\
&\mathsf{project}\,(\Uparrow_{pk}\ d\,y\,f\,p) = \langle \mathsf{project}\,p, y, f\rangle\\
&\mathsf{project}\,(\Downarrow_{pk}\ d\,p) = \mathsf{project}\,p
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{remove} : \{n : \mathbb{N}\}\{d : \mathsf{Dyck}\,n\}(p : \mathsf{Peak}\,d)(\sigma : \mathsf{Sub}) \to \mathsf{Sub}\\
&\mathsf{remove}\,(\Updownarrow_{pk}\ d\,y\,f)\,\langle \sigma, s, t\rangle = \sigma\\
&\mathsf{remove}\,(\Uparrow_{pk}\ d\,y\,f\,p)\,\langle \sigma, s, t\rangle = \langle \mathsf{remove}\,p\,\sigma, s, t\rangle\\
&\mathsf{remove}\,(\Downarrow_{pk}\ d\,p)\,\sigma = \mathsf{remove}\,p\,\sigma
\end{aligned}
$$

Figure 8: Operations on Dyck words

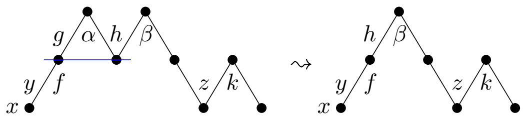


Figure 9: Excising a peak

- *Disc Removal* asserts that unary composites may be removed from the head of a term.
- *Endomorphism Coherence Removal* asserts that coherences associated to a repeated term may be replaced with identities on that term.

We extend this equality relation to types and substitutions by structural induction on the formation rules. It is sufficient to add rules for reflexivity, transitivity, and symmetry to our term equality only; these can then be proven for the other equalities by a simple induction. The remaining rules for our equality are given in Figure 11.

Finally, to integrate our new equality with the type system, we require our term judgement to be equipped with a conversion rule, which allows typing to interact with our equality. This rule is listed in Figure 12.

Example Reductions We record here examples of our three generating reductions, in order to give the reader a flavour of how they operate.

$$
\dfrac{\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma] : B \qquad \alpha : \mathsf{LM}(\Delta) \qquad \alpha[\![\sigma]\!] \equiv \mathbb{1}_{\mathsf{dim}\,\alpha - 1}[\![\tau]\!]}{\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma] = \mathsf{coh}\,(\Delta /\!\!/ \alpha : A[\![\pi_\alpha]\!])[\sigma /\!\!/ x]}\ \textsc{prune}
$$

$$
\dfrac{n : \mathbb{N} \qquad \Gamma \vdash \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\sigma] : B}{\Gamma \vdash \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\sigma] = d_{n+1}[\![\sigma]\!]}\ \textsc{disc}
\qquad
\dfrac{\Gamma \vdash \mathsf{coh}\,(\Delta : t \to_A t)[\sigma] : B}{\Gamma \vdash \mathsf{coh}\,(\Delta : t \to_A t)[\sigma] = \mathbb{1}_{\mathsf{dim}\,A}[\![\{A, t\} \circ \sigma]\!]}\ \textsc{endo}
$$

Figure 10: Generating equality judgements on terms

$$\frac{\Gamma : \mathsf{Ctx} \qquad (x, A) \in \Gamma}{\Gamma \vdash x = x} \qquad \frac{\Delta \vdash A = B \qquad \Gamma \vdash \sigma = \tau}{\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma] = \mathsf{coh}\,(\Delta : B)[\tau]} \qquad \frac{\Gamma \vdash s = t}{\Gamma \vdash t = s}$$

$$\frac{\Gamma \vdash s = t \qquad \Gamma \vdash t = u}{\Gamma \vdash s = u} \qquad \frac{\Gamma : \mathsf{Ctx}}{\Gamma \vdash \star = \star} \qquad \frac{\Gamma \vdash A = A' \qquad \Gamma \vdash s = s' \qquad \Gamma \vdash t = t'}{\Gamma \vdash s \to_A t = s' \to_{A'} t'}$$

$$\frac{\Gamma : \mathsf{Ctx}}{\Gamma \vdash \langle\rangle = \langle\rangle} \qquad \frac{\Gamma \vdash \sigma = \tau \qquad \Gamma \vdash s = t}{\Gamma \vdash \langle \sigma, x \mapsto s\rangle = \langle \tau, x \mapsto t\rangle}$$

Figure 11: Structural equality rules

• **Pruning** If a coherence term has a substitution which sends a locally-maximal variable α to an identity, then the pruning relation allows α to be removed from the corresponding pasting context. Consider the following term:

$$\begin{aligned} &\mathsf{comp}_2[\![x, y, f, y, \mathbb{1}_0[\![y]\!]]\!] \\ &\quad := \mathsf{coh}\,((x : \star)(y : \star)(f : x \to_\star y) \\ &\qquad\qquad (z : \star)(g : y \to_\star z) : x \to_\star z)[x, y, f, y, \mathbb{1}_0[\![y]\!]] \end{aligned}$$

The variable g occurs in a locally maximal position in the head. Moreover, the argument supplied in this position is $\mathbb{1}_0[\![y]\!]$. Hence the pruning relation applies. We have:

$$\begin{aligned} \Gamma /\!\!/ g &:= (x : \star)(y : \star)(f : x \to_\star y) \\ \pi_g &:= [x, y, f, y, \mathbb{1}_0[\![y]\!]] \\ [x, y, f, y, \mathbb{1}_0[\![y]\!]] /\!\!/ g &:= [x, y, f] \end{aligned}$$

so that this term is definitionally equal to the following:

$$\mathsf{coh}\,((x : \star)(y : \star)(f : x \to_\star y) : x \to_\star y)[x, y, f]$$

In more standard notation, we have shown $f \cdot \mathrm{id}_x = (f)$, where (f) represents the unary composite of f.

• **Disc Removal** This relation says that when we are composing over a disc context, with the sphere type, the entire term is definitionally equal to its last argument. This case applies for the unary composite (f) obtained just above. In this case, application of the Disc rule now yields:

$$\mathsf{coh}\,((x : \star)(y : \star)(f : x \to_\star y) : x \to_\star y)[x, y, f] = f$$

Note that this replaces the coherence term with the final argument, f, of its substitution $[x, y, f]$. As f is now a variable, the term is now in normal form (see Section 3).

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t : B}$$

Figure 12: Conversion rule

• **Endomorphism Coherence Removal** A curious redundancy of the fully weak definition of ∞-category is the existence of "fake identities": cells which are "morally" the identity on some composite cell, but do not have an identity coherence at their head. As an example, consider the term:

$$\begin{aligned}&\mathsf{coh}\,((x:\star)(y:\star)(f:x\to_\star y)(z:\star)(g:y\to_\star z)\\ &\quad :\mathsf{comp_2}[\![x,y,f,z,g]\!]\to_{x\to_\star z}\mathsf{comp_2}[\![x,y,f,z,g]\!])\\ &\quad [x,y,f,z,g]\end{aligned}$$

This term is "trying" to be the identity on $\mathsf{comp_2}[\![x,y,f,z,g]\!]$ (indeed, it is provably equivalent to it in Catt), but is not a *syntactic* identity. Such terms are recognizable as coherences for which the type expression has a source and target which are equal. We refer to them as *endomorphism coherences*, and our third rule ENDO sets them equal to the identities they duplicate. In this case, the above term will therefore be definitionally equal to the following:

$$\mathbb{1}_1[\![x,z,\mathsf{comp_2}[\![x,y,f,z,g]\!]]\!]$$

This ENDO reduction has no apparent analog in the theory of Batanin, Cisinski and Weber [5], and we believe in that theory the two terms above would not be identified.

Properties of Definitional Equality In this section, we record two theorems. The first states that equality interacts nicely with composition of substitutions, which will be crucial when we build the syntactic category in order to define models in Section 2.3.

Theorem 3. *If $\sigma=\sigma'$ and $\tau=\tau'$, then $\sigma\circ\tau=\sigma'\circ\tau'$.*

The second is the following major structural theorem, showing that our definitional equality behaves well with typing.

Theorem 4. *Typing is fully compatible with definitional equality, such that the following rules are derivable:*

$$\frac{\Gamma\vdash s:A\qquad\Gamma\vdash s=t\qquad\Gamma\vdash A=B}{\Gamma\vdash t:B}\qquad\frac{\Gamma\vdash A\qquad\Gamma\vdash A=B}{\Gamma\vdash B}\qquad\frac{\Gamma\vdash\sigma:\Delta\qquad\Gamma\vdash\sigma=\tau}{\Gamma\vdash\tau:\Delta}$$

The rest of this section builds the theorem required to prove these theorems. We start by showing that identities and inferred types are valid:

Lemma 5. *Suppose $\Gamma\vdash\{A,t\}:D^k$. Then $\Gamma\vdash t:A$. Conversely, if $\Gamma\vdash t:A$ then $\Gamma\vdash\{A,t\}:D^k$. Further if $\Gamma\vdash t:A$ then $\Gamma\vdash\mathbb{1}_{\mathsf{dim}\,A}[\![\{A,t\}]\!]:t\to_A t$.*

Proof. The proof is simply a rearrangement of some typing data. □

Proposition 6. *If $\Delta\vdash t:A$, then $A=\mathsf{ty}(t)$ and $\Delta\vdash t:\mathsf{ty}(t)$.*

Proof. We induct on the structure of the derivation of $\Delta\vdash t:A$. There are three cases. If the derivation is simply the introduction rule for coherences, then the result is immediate, as $\mathsf{ty}(t)$ is the assigned type. Similarly, if the derivation is the introduction rule for variables, then $\mathsf{ty}(t)$ is again the assigned type.

Otherwise, the proof must be by the use of the conversion rule. In this case, we have that $\Delta\vdash t:B$ and $B=A$. By induction hypothesis we have that $B=\mathsf{ty}(t)$ and so by transitivity and symmetry of equality we get that $A=B=\mathsf{ty}(t)$ as required.

The second part follows from the first by an application of the conversion rule. □

To progress towards a proof of Theorems 3 and 4, we must show that our equality is well behaved. In particular we need to show that it is well behaved with respect to context extension and substitution.

Lemma 7. *Suppose $\Gamma \vdash A$. If $\Gamma \vdash s = t$ then $\Gamma, A \vdash s = t$. Similar results hold for equality of types and substitutions. Further, if $\Gamma \vdash s : B$ then $\Gamma, A \vdash s : B$. Again similar results hold for typing of types and substitutions.*

Proof. All the results are proved by a simple mutual induction. □

To show that the equality interacts well with substitution, we want that substituting a term or type by equal substitutions produces equal terms (or types) and that the action of substitution preserves equality. The first is a simple induction.

Lemma 8. *Let σ, σ' : Sub such that $\Gamma \vdash \sigma = \sigma'$. For A : Type, t : Term, and τ : Sub, we have:*

$$\Gamma \vdash A[\![\sigma]\!] = A[\![\sigma']\!] \quad \Gamma \vdash t[\![\sigma]\!] = t[\![\sigma']\!] \quad \Gamma \vdash \tau \circ \sigma = \tau \circ \sigma'$$

Proof. Since substitution on types is given by structural induction (and the base case $A = \star$ is trivial), the first equation follows from the second. Similarly, the third equation also follows from the second.

Now, for the second, if t is a variable, then the result is clear by the definition of equality on substitutions, which is just equality of the comprising terms. On the other hand, $t \equiv \mathsf{coh}\,(\Gamma : A)[\tau]$, then we are reduced to showing that $\tau \circ \sigma = \tau \circ \sigma'$ which follows by the induction hypothesis (using the third equation). □

To prove that the action of substitution preserves equality we need the following lemma:

Lemma 9. *Composition of substitutions is compatible with taking quotients:*

$$(\mu \circ \sigma) /\!\!/ \alpha \equiv (\mu /\!\!/ \alpha) \circ \sigma$$

Proof. Pruning α from the substitution simply removes two terms. It does not matter if we apply σ to these terms and then remove them, or simply remove them first. □

Lemma 10. *Let A, A' : Type, t, t' : Term, and σ, σ' : Sub. Now suppose we have τ : Sub with $\Gamma \vdash \tau : \Delta$. Then the following implications hold.*

$$\begin{array}{lcl} \Delta \vdash A = A' & \Longrightarrow & \Gamma \vdash A[\![\tau]\!] = A'[\![\tau]\!] \\ \Delta \vdash t = t' & \Longrightarrow & \Gamma \vdash t[\![\tau]\!] = t'[\![\tau]\!] \\ \Delta \vdash \sigma = \sigma' & \Longrightarrow & \Gamma \vdash \sigma \circ \tau = \sigma' \circ \tau \\ \Delta \vdash A & \Longrightarrow & \Gamma \vdash A[\![\tau]\!] \\ \Delta \vdash t : A & \Longrightarrow & \Gamma \vdash t[\![\tau]\!] : A[\![\tau]\!] \\ \Delta \vdash \sigma : \Theta & \Longrightarrow & \Gamma \vdash \sigma \circ \tau : \Theta \end{array}$$

Proof. All parts of the proof follow by mutual induction on all 6 statements. The only difficult cases are those for the second statement, where the equation $t = t'$ is one of the generators of the equality.

Hence, suppose we have

$$\Delta \vdash t \equiv \mathsf{coh}\,(\Gamma : A)[\sigma] = \mathsf{coh}\,(\Gamma /\!\!/ \alpha : A[\![\pi_\alpha]\!])[\sigma /\!\!/ x] \equiv t'$$

Then we are reduced to showing that $\Gamma \vdash (\sigma \circ \tau)/\!\!/\alpha = (\sigma/\!\!/\alpha) \circ \tau$ which follows from Lemma 9.

Next, if we have

$$\Delta \vdash \mathsf{t} \equiv \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\sigma] = d_{n+1}[\![\sigma]\!] \equiv t'$$

we argue as follows:

$$\begin{aligned}\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\sigma][\![\tau]\!] &\equiv \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\sigma \circ \tau] \\ &= d_{n+1}[\![\sigma \circ \tau]\!] \\ &\equiv d_{n+1}[\![\sigma]\!][\![\tau]\!]\end{aligned}$$

Finally, in the case that

$$t \equiv \mathsf{coh}\,(\Gamma : t \to_A t)[\sigma] = \mathbb{1}_{\mathsf{dim}\,A+1}[\![\{A, t\} \circ \sigma]\!] \equiv t'$$

we obtain:

$$\begin{aligned}\mathsf{coh}\,(\Gamma : t \to_A t)[\sigma][\![\tau]\!] &\equiv \mathsf{coh}\,(\Gamma : t \to_A t)[\sigma \circ \tau] \\ &= \mathbb{1}_{\mathsf{dim}\,A+1}[\![\{A, t\} \circ \sigma \circ \tau]\!] \\ &\equiv \mathbb{1}_{\mathsf{dim}\,A+1}[\![\{A, t\} \circ \sigma]\!][\![\tau]\!]\end{aligned}$$

where all syntactic steps follow from Proposition 1. □

We now have enough to prove the first theorem:

Theorem 3. *If $\sigma = \sigma'$ and $\tau = \tau'$, then $\sigma \circ \tau = \sigma' \circ \tau'$.*

Proof. This result is a corollary of Lemmas 8 and 10. □

Next we need to show that the constructions used in pruning are well formed.

Lemma 11. *Suppose $\Delta \vdash_p$, $\Gamma \vdash \sigma : \Delta$, and $\alpha \in \mathsf{LM}(\Delta)$, with $\alpha[\![\sigma]\!] \equiv \mathbb{1}_k[\![\tau]\!]$. Then we have $\Delta/\!\!/\alpha \vdash \pi_\alpha : \Delta$, $\Gamma \vdash \sigma/\!\!/\alpha : \Delta/\!\!/\alpha$, and $\Gamma \vdash \sigma = \pi_\alpha \circ \sigma/\!\!/\alpha$.*

Proof. The proof proceeds by induction on the peak p_α. We begin with π_α. To make notation simpler we will prove for any $d \in \mathsf{Dyck}\,n$ and $p \in \mathsf{Peak}\,d$ that $\lceil \mathsf{excise}\,p \rceil \vdash \mathsf{project}\,p : \lceil d \rceil$, and proceed by induction on p:

In the case that the peak is $\Updownarrow_{\mathsf{pk}}\, d\, y\, f$, then the validity of the substitution

$$\langle \mathrm{id}_{\lceil d \rceil}, \mathsf{tm}(d), \mathbb{1}_{\mathsf{dim}\,\mathsf{ty}(d)}[\![\{\mathsf{ty}(d), \mathsf{tm}(d)\}]\!] \rangle$$

follows by the validity of identity substitutions, validity of $\mathsf{tm}(d)$ and $\mathsf{ty}(d)$ (for which we omit the proof), validity of identities (Lemma 5), and that weakening preserves validity (Lemma 7).

If the peak is of the form $\Uparrow_{\mathsf{pk}}\, d\, y\, f\, p$, then we need to show that

$$\langle \mathsf{project}\,p, y, f \rangle$$

is valid. This follows from weakening, inductive hypothesis and that $\mathsf{ty}(\mathsf{excise}\,p) \equiv \mathsf{ty}(d)[\![\mathsf{project}\,p]\!]$ holds for all d and p (proof of this is again omitted).

If the peak is of the form $\Downarrow_{\mathsf{pk}}\, d\, p$ then we are done by inductive hypothesis.

We now move onto showing that for all $d \in \mathsf{Dyck}\, n$, $p \in \mathsf{Peak}\, d$, and valid $\sigma : \lceil d \rceil \to \Gamma$ that $\Gamma \vdash \mathsf{remove}\, p\, \sigma : \lceil \mathsf{excise}\, p \rceil$ and $\Gamma \vdash \mathsf{project}\, p \circ \mathsf{remove}\, p\, \sigma = \sigma$. We prove these statements by induction on the peak p:

- When the peak is $\Updownarrow_{\mathsf{pk}}\; d\, y\, f$ then the first statement is clear. Suppose we have $\sigma \equiv \langle \tau, s, t \rangle$. For the second we need to show that:

$$\langle \mathrm{id}_{\lceil d \rceil}, \mathsf{tm}(d), \mathbb{1}_{\mathsf{dim}\,\mathsf{ty}(d)}[\![\{\mathsf{ty}(d), \mathsf{tm}(d)\}]\!] \rangle \circ \tau = \langle \tau, s, t \rangle$$

By Proposition 1, $\mathrm{id}_{\lceil d \rceil} \circ \tau \equiv \tau$ so we are left to prove that $\mathsf{tm}(d)[\![\tau]\!] = s$ and $\mathbb{1}_{\mathsf{dim}\,\mathsf{ty}(d)}[\![\{\mathsf{ty}(d), \mathsf{tm}(d)\}]\!][\![\tau]\!] = t$. We now use that $t \equiv f[\![\sigma]\!] \equiv \alpha[\![\sigma]\!] \equiv \mathbb{1}_{\mathsf{dim}\,B}[\![\{B, u\}]\!]$ for some term u and type B.

By Lemma 5 and Proposition 6, we know that $\mathbb{1}_{\mathsf{dim}(B)}[\![\{B, u\}]\!]$ can be given type $u \to_B u$. Further we know that t has type $(\mathsf{tm}(d) \to_{\mathsf{ty}(d)} y)[\![\sigma]\!]$. As t and $\mathbb{1}_{\mathsf{dim}(B)}[\![\{B, u\}]\!]$ are the same term we must have that $s \equiv y[\![\sigma]\!] = u$, $\mathsf{tm}(d)[\![\sigma]\!] = u$, and $\mathsf{ty}(d)[\![\sigma]\!] = B$. Therefore we immediately get that $\mathsf{tm}(d)[\![\tau]\!] \equiv \mathsf{tm}(d)[\![\sigma]\!] = u = s$. Further we have that:

$$\begin{aligned} t &\equiv \mathbb{1}_{\mathsf{dim}(B)}[\![\{B, u\}]\!] \\ &= \mathbb{1}_{\mathsf{dim}(\mathsf{ty}(d))}[\![\{\mathsf{ty}(d)[\![\sigma]\!], \mathsf{tm}(d)[\![\sigma]\!]\}]\!] \\ &\equiv \mathbb{1}_{\mathsf{dim}(\mathsf{ty}(d))}[\![\{\mathsf{ty}(d), \mathsf{tm}(d)\}]\!][\![\sigma]\!] \\ &\equiv \mathbb{1}_{\mathsf{dim}(\mathsf{ty}(d))}[\![\{\mathsf{ty}(d), \mathsf{tm}(d)\}]\!][\![\tau]\!] \end{aligned}$$

as required. The second syntactic equality above comes from Lemma 2.

- In the case that the peak is $\Uparrow_{\mathsf{pk}}\; d\, y\, f\, p$, the second statement follows easily from weakening and inductive hypothesis. For this proof we introduce the shorthand $d /\!/ p := \mathsf{excise}\, p$. Then, for the first we again assume that $\sigma \equiv \langle \tau, s, t \rangle$, which means we need to show that (unwrapping some definitions):

$$\begin{aligned} \Gamma \vdash \langle \mathsf{remove}\, p\, \tau, s, t \rangle : \lceil d /\!/ p \rceil, &(y : \mathsf{ty}(d /\!/ p)), \\ &(f : \mathsf{tm}(d /\!/ p) \to_{\mathsf{ty}(d /\!/ p)} y) \end{aligned}$$

As we have $\Gamma \vdash \langle \tau, s, t \rangle : \lceil d \rceil, (y : \mathsf{ty}(d)), (f : \mathsf{tm}(d) \to_{\mathsf{ty}(d)} y)$ we obtain:

  - Firstly that $\Gamma \vdash \tau : \lceil d \rceil$, which by inductive hypothesis implies that $\Gamma \vdash \mathsf{remove}\, p\, \tau : \lceil d /\!/ p \rceil$.

  - Secondly that $\Gamma \vdash s : \mathsf{ty}(d)[\![\tau]\!]$. From this we prove using the inductive hypothesis and Lemma 8:

$$\begin{aligned} \mathsf{ty}(d)[\![\tau]\!] &= \mathsf{ty}(d)[\![\mathsf{project}\, p \circ \mathsf{remove}\, p\, \tau]\!] \\ &\equiv \mathsf{ty}(d)[\![\mathsf{project}\, p]\!][\![\mathsf{remove}\, p\, \tau]\!] \\ &\equiv \mathsf{ty}(d /\!/ p)[\![\mathsf{remove}\, p\, \tau]\!] \end{aligned}$$

  and so by the conversion rule we have that $\Gamma \vdash s : \mathsf{ty}(d /\!/ p)[\![\mathsf{remove}\, p\, \tau]\!]$.

  - Lastly that $\Gamma \vdash t : (\mathsf{tm}(d) \to_{\mathsf{ty}(d)} y)[\![\langle \tau, s \rangle]\!]$. By simplification,

$$(\mathsf{tm}(d) \to_{\mathsf{ty}(d)} y)[\![\langle \tau, s \rangle]\!] \equiv \mathsf{tm}(d)[\![\tau]\!] \to_{\mathsf{ty}(d)[\tau]} s$$

  As in the last case, $\mathsf{ty}(d)[\![\tau]\!] = \mathsf{ty}(d /\!/ p)[\![\mathsf{remove}\, p\, \tau]\!]$. Similarly $\mathsf{tm}(d)[\![\tau]\!] = \mathsf{tm}(d /\!/ p)[\![\mathsf{remove}\, p\, \tau]\!]$. Lastly we have $s \equiv y[\![\langle \mathsf{remove}\, p\, \tau, s \rangle]\!]$ and so with all these parts and the conversion rule we get that:

$$\Gamma \vdash t : (\mathsf{tm}(d /\!/ p) \to_{\mathsf{ty}(d /\!/ p)} y)[\![\langle \mathsf{remove}\, p\, \tau, s \rangle]\!]$$

By combining all these parts, we have proven what we needed to prove.

- If the peak is of the form $\Downarrow_{\mathsf{pk}}\, d\, p$ then both statements follow from inductive hypothesis.

This covers all of the cases and so all three statements in the lemma hold. $\square$

The last component needed for the proof of Theorem 4 is that support conditions are preserved by definitional equality. For this we need the following lemma.

Lemma 12. *If* $\Gamma \vdash s = t$*, then* $\mathsf{supp}(s) = \mathsf{supp}(t)$*.*

Proof. Define a new set of support-preserving typing judgements $\vdash_s$, which have the same structural and conversion rules (Figures 11 and 12) as $\mathsf{Catt_{su}}$, but for each generator (Figure 10) we add the condition that the terms on the left and right hand side of the equality have the same support. For example, we will have this modified rule for disc removal:

$$\frac{n : \mathbb{N} \qquad \Gamma \vdash_s \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\sigma] : B \qquad \mathsf{supp}(\mathsf{coh}\,(D^{n+1} : S^n)[\sigma]) = \mathsf{supp}(d_{n+1}\llbracket\sigma\rrbracket)}{\Gamma \vdash_s \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\sigma] = d_{n+1}\llbracket\sigma\rrbracket}\ \text{DISC'}$$

By a simple induction, if $\Gamma \vdash_s s = t$ then $\mathsf{supp}(s) = \mathsf{supp}(t)$. In the rest of the proof we prove that the typing judgements $\vdash$ and $\vdash_s$ are actually the same.

It is clear that if a statement is valid under the support-preserving judgement, then it is also valid with the original judgement. Hence we want to show that validity under the original judgements implies validity under the support-preserving judgements. We proceed by a mutual induction on all (original) judgements. Every case is trivial apart from $\Gamma \vdash s = t \implies \Gamma \vdash_s s = t$ where the equality arises from some generator. We proceed through these in turn.

- Suppose we have $\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma] = \mathsf{coh}\,(\Delta /\!/ \alpha : A\llbracket\pi_\alpha\rrbracket)[\sigma /\!/ \alpha]$ from pruning. By assumption we must have $\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma] : B$ for some B, and so by inductive hypothesis $\Gamma \vdash_s \mathsf{coh}\,(\Delta : A)[\sigma] : B$ holds and $\Gamma \vdash_s \sigma : \Delta$. Now note that the proof of Lemma 11 only used Lemmas 7 and 8, for which the only property we required of the generators is that they were preserved by weakening. As support is preserved by weakening, we can deduce that this lemma holds for the support-preserving judgements. This means that $\Gamma \vdash_s \sigma = \pi_\alpha \circ \sigma /\!/ \alpha$ and so $\mathsf{supp}(\sigma) = \mathsf{supp}(\pi_\alpha \circ \sigma /\!/ \alpha)$. As π_α is full we obtain:

$$\begin{aligned}
\mathsf{supp}(\mathsf{coh}\,(\Delta : A)[\sigma]) &= \mathsf{supp}(\sigma)\\
&= \mathsf{supp}(\pi_\alpha \circ \sigma /\!/ \alpha)\\
&= \mathsf{supp}(\sigma /\!/ \alpha)\\
&= \mathsf{supp}(\mathsf{coh}\,(\Delta /\!/ \alpha : A\llbracket\pi_\alpha\rrbracket)[\sigma /\!/ \alpha])
\end{aligned}$$

and so $\Gamma \vdash_s \mathsf{coh}\,(\Delta : A)[\sigma] = \mathsf{coh}\,(\Delta /\!/ \alpha : A\llbracket\pi_\alpha\rrbracket)[\sigma /\!/ \alpha]$ as required.

- Let $\Gamma \vdash \mathsf{coh}\,(D^{n+1} : S^n)[\sigma] = d_{n+1}\llbracket\sigma\rrbracket$ be an instance of disc removal. As in the last case, $\Gamma \vdash_s \sigma : D^{n+1}$. As the notion of support behaves nicely in terms valid in the support-preserving judgements, we get that the support of σ is the same as the support of its top dimensional term $d_{n+1}\llbracket\sigma\rrbracket$, as we would expect. Therefore $\Gamma \vdash_s \mathsf{coh}\,(D^{n+1} : S^n)[\sigma] = d_{n+1}\llbracket\sigma\rrbracket$.
- Now if $\Gamma \vdash \mathsf{coh}\,(\Delta : s \to_A s)[\sigma] = \mathbb{1}_{\mathsf{dim}(A)}\llbracket\{A, t\} \circ \sigma\rrbracket$ is an instance of endo-coherence removal, we once again have that $\Gamma \vdash_s \mathsf{coh}\,(\Delta : s \to_A s)[\sigma] : B$ for some type B. By the typing rules for coherences, we must either have that $\mathsf{supp}(s) = \partial^-(\Gamma)$ and

$\mathsf{supp}(s) = \partial^+(\Gamma)$, or that $\mathsf{supp}(s) = \mathsf{FV}(\Gamma)$. As $\partial^-(\Gamma)$ and $\partial^+(\Gamma)$ are not equal, we deduce that the second condition must hold, and so $\{A, s\}$ is full. Therefore:

$$\begin{aligned}\mathsf{supp}(\mathsf{coh}\,(\Delta : s \to_A s)[\sigma]) &= \mathsf{supp}(\sigma)\\ &= \mathsf{supp}(\{A, s\} \circ \sigma)\\ &= \mathsf{supp}(\mathbb{1}_{\mathsf{dim}(A)}[\![\{A, s\} \circ \sigma]\!])\end{aligned}$$

and so $\Gamma \vdash_s \mathsf{coh}\,(\Delta : s \to_A s)[\sigma] = \mathbb{1}_{\mathsf{dim}(A)}[\![\{A, s\} \circ \sigma]\!]$.

Finally, since we have proven that $\Gamma \vdash s = t$ implies $\Gamma \vdash_s s = t$, and that $\Gamma \vdash_s s = t$ implies that $\mathsf{supp}(s) = \mathsf{supp}(t)$, we have completed the proof. $\square$

We can then finally prove Theorem 4.

Theorem 4. *Typing is fully compatible with definitional equality, such that the following rules are derivable:*

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash s = t \qquad \Gamma \vdash A = B}{\Gamma \vdash t : B} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash A = B}{\Gamma \vdash B} \qquad \frac{\Gamma \vdash \sigma : \Delta \qquad \Gamma \vdash \sigma = \tau}{\Gamma \vdash \tau : \Delta}$$

Proof of Theorem 4. We proceed by mutual induction on the derivation of the equalities ($\Gamma \vdash s = t$, $\Gamma \vdash A = B$, and $\Gamma \vdash \sigma = \tau$). In fact, it is necessary for our inductive hypothesis to be of the form "if $\Gamma \vdash s = t$ and $\Gamma \vdash A = B$, then $\Gamma \vdash s : A$ *if and only if* $\Gamma \vdash t : B$" and similar for terms and substitutions. The cases for types and substitutions are easy, and so we run through the cases for terms. We note that it is sufficient to prove the statement "if $\Gamma \vdash s = t$ then s is valid if and only if t is valid and $\mathsf{ty}(s) = \mathsf{ty}(t)$", from which we can recover the original statement using the conversion rule and Proposition 6.

- If the equality is from reflexivity on variables, we have nothing to prove.
- If the equality is from the transitivity constructor, we use that "if and only if" is a transitive relation.
- Similarly if the equality arises from the symmetry constructor, we again use that "if and only if" is a symmetric relation. Note that this is why we couldn't have used the weaker induction hypothesis of the form "if $\Gamma \vdash s = t$ and s is valid then t is valid".
- For the structural rule for coherences, we have $\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma] = \mathsf{coh}\,(\Delta : B)[\tau]$ from $A = B$ and $\sigma = \tau$. Suppose $\mathsf{coh}\,(\Delta : A)[\sigma]$ is valid. Firstly by inductive hypothesis we have $\Delta \vdash B$ and $\Gamma \vdash \tau : \Delta$. By Lemma 12, we have that the relevant support condition for $\mathsf{coh}\,(\Delta : B)[\tau]$ is satisfied and so this term is valid. The other direction follows symmetrically. Finally we also have:

$$\mathsf{ty}(\mathsf{coh}\,(\Delta : A)[\sigma]) \equiv A[\![\sigma]\!] = B[\![\tau]\!] \equiv \mathsf{ty}(\mathsf{coh}\,(\Delta : B)[\tau])$$

where the equality uses Lemmas 8 and 10.

- Suppose we have $\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma] = \mathsf{coh}\,(\Delta /\!\!/ \alpha : A[\![\pi_\alpha]\!])[\sigma /\!\!/ \alpha]$ from pruning. The backwards direction is clear so we prove the forwards direction. Suppose $\Gamma \vdash \mathsf{coh}\,(\Delta : A)[\sigma]$ is valid, which implies $\Gamma \vdash \sigma : \Delta$ and $\Delta \vdash A$. Therefore by Lemma 11, $\Delta /\!\!/ \alpha \vdash \pi_\alpha : \Delta$ and $\Gamma \vdash \sigma /\!\!/ \alpha : \Delta /\!\!/ \alpha$. By Lemma 10, $\Delta \vdash A[\![\pi_\alpha]\!]$.

All that remains to show that the right hand side of the equation is valid is that the appropriate support condition holds. Assume that $A \equiv s \to_{A'} t$. First, assume that $\mathsf{supp}(s) = \mathsf{supp}(t) = \mathsf{FV}(\Delta)$. Then,

$$\mathsf{supp}(s[\![\pi_\alpha]\!]) = \mathsf{supp}(t[\![\pi_\alpha]\!]) = \mathsf{supp}(\pi_\alpha) = \mathsf{FV}(\Delta /\!\!/ \alpha)$$

Otherwise $\mathsf{supp}(s) = \partial^-(\Delta)$ and $\mathsf{supp}(t) = \partial^+(\Delta)$. We first suppose that the dimension of $\Delta /\!\!/ \alpha$ is one less than the dimension of Δ. In this case, α was the only variable of maximum dimension and so $\mathsf{supp}(s)$ is the whole of Δ except α and $\mathsf{tgt}(\alpha)$ and $\mathsf{supp}(t)$ is the whole of Δ except α and $\mathsf{src}(\alpha)$. By the definition of π_α it is immediate that $\mathsf{supp}(s[\![\pi_\alpha]\!]) = \mathsf{supp}(t[\![\pi_\alpha]\!]) = \mathsf{FV}(\Delta /\!\!/ \alpha)$, which is sufficient for validity.

The last case is as above but assuming that the dimension of Δ and $\Delta /\!\!/ \alpha$ are the same. Then $\partial^-(\Delta /\!\!/ \alpha) = \partial^-(\Delta)$ as they both have all lower dimensional variables, no maximal dimensional variables and all target-free variables, which are the same in both. Since π_α is the identity of $\partial^-(\Delta)$ we have that $\mathsf{supp}(s[\![\pi_\alpha]\!]) = \partial^-(\Delta /\!\!/ \alpha)$. Similarly we have that $\partial^+(\Delta /\!\!/ \alpha) = \partial^+(\Delta) \setminus \{\mathsf{tgt}(\alpha)\} \cup \{\mathsf{src}(\alpha)\}$ by inspecting which variables are source free. Then as π_α is the identity on $\partial^+(\Delta) \setminus \{\mathsf{tgt}(\alpha)\}$ and sends $\mathsf{tgt}(\alpha)$ to $\mathsf{src}(\alpha)$ we have that $\mathsf{supp}(t) = \partial^+(\Delta /\!\!/ \alpha)$ as required.

Finally we have:

$$\begin{aligned}
\mathsf{ty}(\mathsf{coh}\,(\Delta : A)[\sigma]) &\equiv A[\![\sigma]\!] \\
&= A[\![\pi_\alpha \circ \sigma /\!\!/ \alpha]\!] \\
&\equiv A[\![\pi_\alpha]\!][\![\sigma /\!\!/ \alpha]\!] \\
&\equiv \mathsf{ty}(\mathsf{coh}\,(\Delta /\!\!/ \alpha : A[\![\pi_\alpha]\!])[\sigma /\!\!/ \alpha])
\end{aligned}$$

from Lemmas 8 and 11.

• For disc removal, we again only need to show the forward direction. If $\mathsf{coh}\,(\mathrm{D}^{k+1} : S^k)[\sigma]$ is valid, then σ is too. Since σ is a substitution from a disc, $\sigma \equiv \{S^k[\![\sigma]\!], d_{k+1}[\![\sigma]\!]\}$, and by Lemma 5, we have $\Gamma \vdash d_{k+1}[\![\sigma]\!] : S^k[\![\sigma]\!]$. Further, by Proposition 6, we must have $\mathsf{ty}(d_{k+1}[\![\sigma]\!]) = S^k[\![\sigma]\!] \equiv \mathsf{ty}(\mathsf{coh}\,(\mathrm{D}^{k+1} : S^k)[\sigma])$.

• For endo-coherence removal, we once again only consider the forwards direction. If $\mathsf{coh}\,(\Delta : s \to_A s)[\sigma]$ is valid in Γ, then $\Gamma \vdash \sigma : \Delta$ and $\Delta \vdash s : A$. Therefore by Lemma 5 $\Delta \vdash \{A, s\} : \mathrm{D}^{\mathsf{dim}(A)+1}$ and so $\Gamma \vdash \{A, s\} \circ \sigma : \mathrm{D}^{\mathsf{dim}(A)+1}$. This implies that $\mathbb{1}_{\mathsf{dim}(A)}[\![\{A, s\} \circ \sigma]\!]$ is valid and by Lemma 2 we have $\mathbb{1}_{\mathsf{dim}(A)}[\![\{A, s\} \circ \sigma]\!] \equiv \mathbb{1}_{\mathsf{dim}(A)}[\![\{A[\![\sigma]\!], s[\![\sigma]\!]\}]\!]$ and so

$$\begin{aligned}
\mathsf{ty}(\mathsf{coh}\,(\Delta : s \to_A s)[\sigma]) &\equiv s[\![\sigma]\!] \to_{A[\sigma]} s[\![\sigma]\!] \\
&\equiv \mathsf{ty}(\mathbb{1}_{\mathsf{dim}(A)}[\![\{A[\![\sigma]\!], s[\![\sigma]\!]\}]\!]) \\
&\equiv \mathsf{ty}(\mathbb{1}_{\mathsf{dim}(A)}[\![\{A, s\} \circ \sigma]\!])
\end{aligned}$$

which is what we needed to prove.

The induction hypothesis is a strengthening of the statement of the theorem, and so we are done. □

2.3 Models of Catt and $\mathsf{Catt}_{\mathsf{su}}$

The type theories Catt and $\mathsf{Catt}_{\mathsf{su}}$ generate syntactic categories via a standard construction: objects are contexts, and morphisms are substitutions, up to definitional equality. Composition is given by composition of substitutions, which is well-defined (for $\mathsf{Catt}_{\mathsf{su}}$) by Theorem 3, and associative and unital (for both theories) by Proposition 1. We abuse notation slightly and write Catt and $\mathsf{Catt}_{\mathsf{su}}$ for the corresponding categories. We also write $\mathsf{Catt}^{\mathsf{pd}}$ and $\mathsf{Catt}^{\mathsf{pd}}_{\mathsf{su}}$ for the full subcategories consisting of just the pasting contexts.

As we have seen, this category contains a collection of objects D^k corresponding to the k-dimensional disc contexts. Together, these contexts and their source and target substitutions constitute a *globular object* in Catt and $\mathsf{Catt_{su}}$, respectively, i.e. a diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathrm{D}^{k+1} \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathrm{D}^{k} \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \cdots \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathrm{D}^{0}
\end{tikzcd}$$

such that $\sigma \circ \sigma = \sigma \circ \tau$ and $\tau \circ \sigma = \tau \circ \tau$.

Definition 13. A category $\mathcal{C}$ containing a globular object is said to admit *globular limits* if every diagram of the following form admits a limit, where $j_k \leq i_{k-1}, i_k$ for all $1 \leq k \leq n$:

$$\begin{tikzcd}
D^{i_0} \arrow[dr, "\underbrace{\tau \circ \cdots \circ \tau}_{i_0 - j_1}"'] & & D^{i_1} \arrow[dl, "\underbrace{\sigma \circ \cdots \circ \sigma}_{i_1 - j_1}"] & \cdots & & D^{i_n} \arrow[dl, "\underbrace{\sigma \circ \cdots \circ \sigma}_{i_n - j_n}"] \\
& D^{j_1} & & & D^{j_n} &
\end{tikzcd}$$

Dually, a category $\mathcal{C}$ containing a co-globular object is said to admit *globular sums* if the category $\mathcal{C}^{\mathrm{op}}$ admits globular limits.

Theorem 14. Catt *and* $\mathsf{Catt_{su}}$ *admit globular limits.*

The proof that Catt admits globular limits is to some extent folklore, and has recently been written out [6]. The proof for $\mathsf{Catt_{su}}$ is the same, as it depends only on the variable structure of pasting contexts, which is the same in both theories.

With this in hand, we can give our notion of model.

Definition 15. An ∞*-category* is a presheaf on the category $\mathsf{Catt^{pd}}$ which sends globular limits to globular sums. A *strictly unital* ∞*-category* is a presheaf on $\mathsf{Catt^{pd}_{su}}$ which sends globular limits to globular sums.

We write $\mathcal{C}at_\infty$ for the resulting category of ∞-categories and $\mathcal{C}at^{\mathrm{su}}_\infty$ for the strictly unital ∞-categories, each being full subcategories of the category of presheaves on $\mathsf{Catt^{pd}}$ and $\mathsf{Catt^{pd}_{su}}$ respectively.

Note that since every valid Catt term will also be a valid $\mathsf{Catt_{su}}$ term, there is an evident functor $K : \mathsf{Catt^{pd}} \to \mathsf{Catt^{pd}_{su}}$ which is the identity on objects (pasting diagrams are the same in both theories), and which sends substitutions to their equivalence classes under definitional equality. Since K preserves globular limits, precomposition with K determines a functor $K^* : \mathcal{C}at^{\mathrm{su}}_\infty \to \mathcal{C}at_\infty$, sending a strictly unital ∞-category to its underlying ∞-category. We will see in Section 6 that this functor is fully-faithful; that is, that every ∞-category can arise from at most one strictly unital ∞-category in this way. In other words, being strictly unital is a *property* of an ∞-category.

3 Reduction

Overview In this section we introduce a reduction relation $\rightsquigarrow$ on types, terms and substitutions, and show that its reflexive, transitive and symmetric closure agrees with definitional equality. We then define a subrelation called *standard reduction*, written $\boldsymbol{\rightsquigarrow}$, and show it is a partial function which terminates after finitely many steps, giving a notion of normal form. Finally, we show that two terms have the same normal form just when they are definitionally equal, meaning that standard reduction gives an algorithm for deciding definitional equality.

Convention on Contexts We say "U is valid in Γ" to mean $\Gamma \vdash U$. We say "t is valid in Γ" to mean there is some type U such that $\Gamma \vdash t : U$. We say "t is valid" to mean there exists some context Γ and type U such that $\Gamma \vdash t : U$.

When a fixed context Γ is under consideration, we will often write $s = t$ to mean $\Gamma \vdash s = t$, and will do similar for types and substitutions.

Reflexive, Transitive, Symmetric Closures Given a relation $\rightsquigarrow$ between terms, types, or substitutions, we write $\rightsquigarrow_{\mathsf{r}}$ for its reflexive closure, $\rightsquigarrow_{\mathsf{t}}$ for its transitive closure, and $\rightsquigarrow_{\mathsf{s}}$ for its symmetric closure, in the set of valid syntactic entities. When we use multiple such subscripts, we mean this simultaneously; for instance, we write $\rightsquigarrow_{\mathsf{rts}}$ for its simultaneous reflexive, transitive and symmetric closure of valid entities.

3.1 General reduction

We define a reduction relation on types, terms and substitutions, and show that the equivalence relation generated by these relations agrees with definitional equality.

We first define a simple syntactic property on terms, that of being an identity.

Definition 16. A term is an *identity* if it is of the form $\mathbb{1}_n[\![\sigma]\!]$ for some $n \in \mathbb{N}$; that is, when its head is an identity coherence.

We emphasise that as a syntactic property, this is *not* compatible with definitional equality. For example, if $t \equiv \mathbb{1}_n[\![\sigma]\!]$, then t is an identity; but if we merely have $t = \mathbb{1}_n[\![\sigma]\!]$, then t is not necessarily an identity.

We now give the reduction relation. The following definitions are given by simultaneous induction. Determining whether a given type, term or substitution is a redex is a purely *syntactic* condition, which can be mechanically verified, and does not refer to definitional equality.

Definition 17 (Reduction of types)**.** The basic type $\star$ does not reduce. An arrow type $U \equiv (u \to_T v)$ reduces as follows:

(T1) if $u \rightsquigarrow u'$, then: $(u \to_T v) \overset{\text{T1}}{\rightsquigarrow} (u' \to_T v)$

(T2) if $v \rightsquigarrow v'$, then: $(u \to_T v) \overset{\text{T2}}{\rightsquigarrow} (v \to_T v')$

(T3) if $T \rightsquigarrow T'$, then: $(u \to_T v) \overset{\text{T3}}{\rightsquigarrow} (u \to_{T'} v)$

Definition 18 (Reduction of substitutions)**.** A substitution $\sigma \equiv [s_1, \ldots, s_n]$ reduces as follows, given a reduction $s_i \rightsquigarrow s_i'$ of some argument:

$$[s_1, \ldots, s_i, \ldots, s_n] \overset{\text{S}}{\rightsquigarrow} [s_1, \ldots, s_i', \ldots, s_n]$$

Definition 19 (Reduction of terms)**.** Variable terms do not reduce. A coherence $t \equiv \mathsf{coh}\,(\Gamma : T)[\sigma]$ reduces as follows:

(A) if $\sigma \rightsquigarrow \sigma'$, then:

$$\mathsf{coh}\,(\Gamma : T)[\sigma] \overset{\text{A}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : T)[\sigma']$$

(B) if t is not an identity, and $x \in \mathsf{LM}(\Gamma)$ for which $x[\sigma]$ is an identity, then we define:

$$\mathsf{coh}\,(\Gamma : T)[\sigma] \overset{\text{B}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma /\!\!/ x : T[\![\pi_x]\!])[\sigma /\!\!/ x]$$

(C) if $T \rightsquigarrow T'$, then:

$$\mathsf{coh}\,(\Gamma : T)[\sigma] \overset{\mathrm{C}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : T')[\sigma]$$

(D) the disc removal relation:

$$\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, t] \overset{\mathrm{D}}{\rightsquigarrow} t$$

(E) if t is not an identity, the endomorphism coherence removal relation:

$$\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] \overset{\mathrm{E}}{\rightsquigarrow} \mathbb{1}_{\mathsf{dim}\,T+1}[\![\{T, u\} \circ \sigma]\!]$$

If we can reduce $u \rightsquigarrow u'$ via some reduction stage (X) above, we say that u is a *general X-redex*, or just an *X-redex*, and we write $u \overset{\mathrm{X}}{\rightsquigarrow} u'$. A given term can be a general X-redex for multiple stages (X). For example, if $u \rightsquigarrow u'$, then the term $\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma]$ is a C-redex in at least 2 ways, and also an E-redex, as follows:

$$\begin{aligned}
\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] &\overset{\mathrm{C}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : u' \to_T u)[\sigma] \\
\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] &\overset{\mathrm{C}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : u \to_T u')[\sigma] \\
\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] &\overset{\mathrm{E}}{\rightsquigarrow} \mathbb{1}_{\mathsf{dim}\,T+1}[\![\{T, u\} \circ \sigma]\!]
\end{aligned}$$

A term could also have no reductions at all. So reduction is partially defined, and multivalued in general.

We now show that the equivalence relation generated by this reduction relation agrees with definitional equality constructed in Section 2.

Proposition 20. *For any context Θ, and for any terms, types, and substitutions valid in Θ, the equality relation $=$ agrees with $\rightsquigarrow_{\mathsf{rts}}$, where $\rightsquigarrow_{\mathsf{rts}}$ is the reflexive transitive symmetric closure of $\rightsquigarrow$ in the set of terms, types, or substitutions valid in Θ.*

Proof. To prove $\rightsquigarrow_{\mathsf{rts}}$ implies $=$, it is sufficient to prove that $s \rightsquigarrow t$ implies $s = t$ for terms s and t valid in Θ, as $=$ is an equivalence relation and the statement for types and substitutions reduces to the statement for terms. This statement follows from a simple induction, noting that subterms of valid terms are themselves valid.

We say that a term p is *conservative* if for all contexts Θ and terms q such that p and q are valid in Θ, we have that $p = q$ implies $p \rightsquigarrow_{\mathsf{rts}} q$. To prove the lemma, we must therefore show that all terms are conservative. Our proof operates by induction on subterms of p, and by case analysis on the equality $p = q$. Almost all such cases are immediate; here we explicitly handle the only non-trivial case.

Suppose $p = q$ is the following equality, obtained by pruning an identity term:

$$\begin{aligned}
&\mathbb{1}_n[\![\ldots, v_2, v_1, v_1', \mathbb{1}_{n-1}[\![\ldots, u_3, u_2, u_1]\!]]\!] \\
&= (\mathsf{coh}\,\mathrm{D}^{n-1} : \mathbb{1}_{n-1} \to_{(d_{n-1} \to_{S^{n-2}} d_{n-1})} \mathbb{1}_{n-1})[\ldots, v_2, v_1]
\end{aligned}$$

By induction on subterms of p we conclude that the terms u_i, v_1', v_i are conservative. By validity we must have $u_1 = v_1 = v_1'$ and $u_i = v_i$, and hence we conclude $u_1 \rightsquigarrow_{\mathsf{rts}} v_1 \rightsquigarrow_{\mathsf{rts}} v_1'$ and $u_i \rightsquigarrow_{\mathsf{rts}} v_i$. We do not have $p \overset{\mathrm{B}}{\rightsquigarrow} q$, since p is an identity term, which are explicitly proscribed as B-redexes. However, q admits an E-reduction, and then a further series of A-reductions obtained by conservativity of the subterms of p, as follows:

$$\begin{aligned} q &\overset{\text{E}}{\rightsquigarrow} \mathbb{1}_n[\![\dots, v_3, v_2, v_1, v_1, \mathbb{1}_{n-1}[\![\dots, v_3, v_2, v_1]\!]]\!] \\ &\overset{\text{A}}{\rightsquigarrow}_{\mathsf{rts}} \mathbb{1}_n[\![\dots, v_3, v_2, v_1, v_1', \mathbb{1}_{n-1}[\![\dots, u_3, u_2, u_1]\!]]\!] \\ &\equiv p \end{aligned}$$

Hence $q \rightsquigarrow_{\mathsf{rts}} p$ as required. □

From this we get the following lemma as a corollary.

Corollary 21. *Reduction of types, terms and substitutions preserves judgement validity:*

- *if* $\Gamma \vdash A$ *is valid and* $A \rightsquigarrow A'$*, then* $\Gamma \vdash A'$ *is valid;*
- *if* $\Gamma \vdash t : A$ *is valid and* $t \rightsquigarrow t'$*, then* $\Gamma \vdash t' : A$ *is valid;*
- *if* $\Gamma \vdash t : A$ *is valid and* $A \rightsquigarrow A'$*, then* $\Gamma \vdash t : A'$ *is valid;*
- *if* $\Gamma \vdash \sigma : \Delta$ *is valid and* $\sigma \rightsquigarrow \sigma'$*, then* $\Gamma \vdash \sigma' : \Delta$ *is valid.*

Proof. Immediate from Proposition 20 and Theorem 4. □

3.2 Standard reduction

We now define *standard reduction*, denoted by $\boldsymbol{\rightsquigarrow}$, a sub-relation of general reduction $\rightsquigarrow$. Standard reduction is a reduction strategy, in the following sense.

Definition 22. A *reduction strategy* is a relation $\to$ on terms with the property that, if $a \to b$ and $a \to b'$, then $b \equiv b'$.

Standard reduction works in a similar way to general reduction, but the reductions now have a preference order, so that higher-priority redexes, listed earlier in the following list, block lower-priority redexes listed later. Standard reduction is hence a reduction strategy by construction.

Definition 23 (Standard reduction of types)**.** The standard reduction of a type $U \equiv (u \to_T v)$ is given by the first matching reduction in the following list, if any:

(T0) if $T \boldsymbol{\rightsquigarrow} \widetilde{T}$, then: $(u \to_T v) \overset{\text{T0}}{\boldsymbol{\rightsquigarrow}} (u \to_{\widetilde{T}} v)$

(T1) if $u \boldsymbol{\rightsquigarrow} \tilde{u}$, then: $(u \to_T v) \overset{\text{T1}}{\boldsymbol{\rightsquigarrow}} (\tilde{u} \to_T v)$

(T2) if $v \boldsymbol{\rightsquigarrow} \tilde{v}$, then: $(u \to_T v) \overset{\text{T2}}{\boldsymbol{\rightsquigarrow}} (v \to_T \tilde{v})$

Definition 24 (Standard reduction of substitutions)**.** Given a substitution $\sigma \equiv \langle s_1, \dots, s_n \rangle$, then if $s_i \boldsymbol{\rightsquigarrow} \tilde{s}_i$ is the leftmost argument with a standard reduction, we have the following:

$$[s_1, \dots, s_i, \dots, s_n] \overset{\text{S}}{\boldsymbol{\rightsquigarrow}} [s_1, \dots, \tilde{s}_i, \dots, s_n]$$

Definition 25 (Standard reduction of terms)**.** A coherence term $t \equiv \mathsf{coh}\,(\Gamma : U)[\sigma]$ has a *standard reduction* given by the first listed reduction which is defined, if any:

(A) if $\sigma \rightsquigarrow \tilde{\sigma}$, then:

$$\mathsf{coh}\,(\Gamma : U)[\sigma] \overset{\mathrm{A}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : U)[\tilde{\sigma}]$$

(B) if t is not an identity, and $x \in \mathsf{var}(\Gamma)$ is the leftmost locally-maximal variable for which $x[\sigma]$ is an identity, then we define:

$$\mathsf{coh}\,(\Gamma : U)[\sigma] \overset{\mathrm{B}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma /\!\!/ x : U[\![\pi_x]\!])[\sigma /\!\!/ x]$$

(C) if $T \rightsquigarrow \tilde{T}$, then:

$$\mathsf{coh}\,(\Gamma : T)[\sigma] \overset{\mathrm{C}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : \tilde{T})[\sigma]$$

(D) the disc removal relation:

$$\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, t] \overset{\mathrm{D}}{\rightsquigarrow} t$$

(E) if t is not an identity, the endo-coherence removal relation:

$$\mathsf{coh}\,(\Gamma : u \to_A u)[\sigma] \overset{\mathrm{E}}{\rightsquigarrow} \mathbb{1}_{\mathsf{dim}\,A+1}[\![\{A, u\} \circ \sigma]\!]$$

If we can reduce $s \rightsquigarrow t$ via some reduction label (X) above, we say that u is a *standard X-redex*. It is an immediate consequence of the definition of standard reduction that it is a reduction strategy; that is, if a term, type or substitution has a standard reduction, it has *exactly one* standard reduction. This is quite unlike general reduction as defined as above. For example, suppose $u \rightsquigarrow \tilde{u}$, and consider the term $t \equiv \mathsf{coh}\,(\Gamma : u \to_A u)[\sigma]$. It is possible that t is a standard A-redex; failing that, it could be a standard B-redex; failing that, it will certainly be a standard C-redex. Although t is an E-redex (that is, there exists t' with $t \overset{\mathrm{E}}{\rightsquigarrow} t'$), it is not a *standard* E-redex, since standard C-reductions are higher-priority than standard E-reductions.

Since standard reduction is unique when it exists, it is useful to introduce the following notation.

Definition 26. If s has a standard reduction, we write it as $\tilde{s}$, and hence $s \rightsquigarrow \tilde{s}$. We call $\tilde{s}$ the *standard reduct* of s.

These simple lemmas now follow.

Lemma 27. *If $s \rightsquigarrow \tilde{s}$, then $s \rightsquigarrow \tilde{s}$.*

Proof. By definition, standard reduction is a subrelation of general reduction. □

Lemma 28. *If $s \rightsquigarrow t$, then there is a unique $\tilde{s}$ with $s \rightsquigarrow \tilde{s}$.*

Proof. Since standard reduction is a reduction strategy, uniqueness is clear. What we must establish is existence. The intuition is straightforward: in essence, we define standard reduction by giving a priority order to the redexes for general reduction, and allowing only the highest-priority redex. The result is then immediate, because if s has at least one reduction, then there must be a highest-priority such reduction.

We prove the result formally as follows, by simultaneous induction on the structure of terms, types, and substitutions. For the base cases, given by the type $\star$ or a variable term, there is no reduction, so the claim is vacuously true.

For a compound type $U \equiv (a \to_T b)$ is a type, then the statement follows immediately by induction on a, b or T.

For a substitution $\sigma = \langle s_1, \ldots, s_n \rangle$, suppose we have some reduction $\sigma \rightsquigarrow \tau$ arising from some choice of index i and some reduction $s_i \rightsquigarrow t$. Since σ has a reducible argument, it must have a leftmost reducible argument, which we can write as s_j, with $j \leq i$. By induction on subterms $s_j \boldsymbol{\rightsquigarrow} \tilde{s}_j$, and $[\ldots, s_{j-1}, s_j, s_{j+1}, \ldots] \boldsymbol{\rightsquigarrow} [\ldots, s_{j-1}, \tilde{s}_j, s_{j+1}, \ldots]$ is the required standard reduction.

For a coherence term $s \equiv \mathsf{coh}\,(\Gamma : T)[\sigma]$ with a reduction $s \rightsquigarrow t$, we argue by case analysis as follows.

- If u is an A-redex, there must exist some σ' such that $\sigma \rightsquigarrow \sigma'$. By induction $\sigma \boldsymbol{\rightsquigarrow} \tilde{\sigma}$, and hence $\mathsf{coh}\,(\Gamma : T)[\sigma] \overset{\mathrm{A}}{\boldsymbol{\rightsquigarrow}} \mathsf{coh}\,(\Gamma : T)[\tilde{\sigma}]$.
- If u is not an A-redex, but u is a B-redex, then there must be some leftmost locally-maximal argument of Γ with respect to which it is a standard B-redex.
- If u is not an A- or B-redex, but it is a C-redex, then there must exist some T' such that $T \rightsquigarrow T'$. Hence by induction $T \boldsymbol{\rightsquigarrow} \tilde{T}$, and so $u \overset{\mathrm{C}}{\boldsymbol{\rightsquigarrow}} \mathsf{coh}\,(\Gamma : \tilde{T})[\sigma]$.
- If u is not an A-, B- or C-redex, but it is a D-redex, then the D-reduction will be standard.
- If u is not an A-, B-, C- or D-redex, but it is an E-redex, then the E-reduction will be standard.
- If u is not an A-, B-, C-, D- or E-redex, then u cannot be reduced, contradicting the hypothesis of the theorem.

This completes the argument. □

3.3 Termination of standard reduction

Standard reduction gives us a notion of normal form.

Definition 29. We define a term, type or substitution to be in *normal form* when it has no reduction, either by standard or general reduction (by Lemmas 27 and 28, these conditions are equivalent.)

In this section we show that standard reduction terminates after finite time on every term, type and substitution. This means that for every term t we can obtain a term $N(t)$ in normal form, by repeatedly applying standard reduction until a normal form is reached.

To work towards our termination result, we consider what happens when we start with a given term and repeatedly perform standard reduction. What we find is a specific pattern of standard reductions, illustrated in Figure 13. We first perform some family of standard A-reductions. If these terminate, they will be followed by some family of standard B-reductions. If these terminate, they will be followed by some family of standard C-reductions. If these terminate, and they are not yet sufficient to yield a normal form, they will be followed either by a single standard D-reduction, giving a term in normal

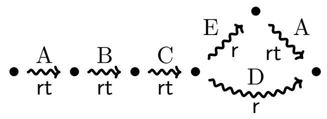


Figure 13: Standard reduction pathways to normal form

form; or by a single standard E-reduction and a family of standard A-reductions, which if they terminate will yield a normal form. We prove this claim here.

Proposition 30. *The reflexive transitive closure of the standard reduction relation is obtained as the following composite:*

$$\rightsquigarrow_{\mathsf{rt}} \quad = \quad \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} \overset{\mathrm{B}}{\rightsquigarrow}_{\mathsf{rt}} \overset{\mathrm{C}}{\rightsquigarrow}_{\mathsf{rt}} \left(\overset{\mathrm{D}}{\rightsquigarrow}_{\mathsf{r}} \cup \overset{\mathrm{E}}{\rightsquigarrow}_{\mathsf{r}} \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} \right)$$

Proof. We establish the claim by showing that the following composite reduction pairs and triples are impossible (that is, they are empty as relations).

- $p \overset{\mathrm{B}}{\rightsquigarrow} q \overset{\mathrm{A}}{\rightsquigarrow} r$. For p to be a standard B–redex, it is required that it is not an A-redex, and so the arguments of p must be in normal form. But the arguments of q are a subset of the arguments of p, contradicting the claim that q is a standard A-redex.

- $p \overset{\mathrm{C}}{\rightsquigarrow} q \overset{\mathrm{A}}{\rightsquigarrow} r$. For p to be a standard C-redex, it is required that it is not an A-redex, and so the arguments of p must be in normal form. But C-reductions do not change the arguments of a term, so the arguments of q are the same as the arguments p, contradicting the claim that q is a standard A-redex.

- $p \overset{\mathrm{C}}{\rightsquigarrow} q \overset{\mathrm{B}}{\rightsquigarrow} r$. For p to be a standard C-redex, it is required that it is not a B-redex, a condition that depends on the context and arguments of the term. But C-reductions do not change the context or arguments, so it is impossible that q is a B-redex.

- $p \overset{\mathrm{D}}{\rightsquigarrow} q \rightsquigarrow r$. For a term to be a standard D-redex, it is required that it is not an A-redex, and so the arguments of p must not be reducible. By the action of D-reduction, the term q is one of the arguments of p, contradicting the claim that q is reducible.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{B}}{\rightsquigarrow} r$. The standard E-reduct q is an identity term, but identity terms are never B-redexes, by definition.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{C}}{\rightsquigarrow} r$. The standard E-reduct q is an identity term, but identity terms are never C-redexes, since the cell part is in normal form.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{D}}{\rightsquigarrow} r$. The standard E-reduct q is an identity term, but identity terms are never D-redexes, as the head has the wrong syntactic form.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{E}}{\rightsquigarrow} r$. The standard E-reduct q is an identity term, but identity terms are never E-redexes, by definition.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} r \overset{\mathrm{B}}{\rightsquigarrow} s$. The standard E-reduct q is an identity term. A-reductions do not change the head, and so r will also be an identity term. But identity terms are never B-redexes, giving a contradiction.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} r \overset{\mathrm{C}}{\rightsquigarrow} s$. The standard E-reduct q is an identity term. A-reductions do not change the head, and so r will also be an identity term. But identity terms are never C-redexes, giving a contradiction.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} r \overset{\mathrm{D}}{\rightsquigarrow} s$. The standard E-reduct q is an identity term. A-reductions do not change the head, and so r will also be an identity term. But identity terms are never D-redexes, giving a contradiction.

- $p \overset{\mathrm{E}}{\rightsquigarrow} q \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} r \overset{\mathrm{E}}{\rightsquigarrow} s$. The standard E-reduct q is an identity term. A-reductions do not change the head, and so r will also be an identity term. But identity terms are never E-redexes, by definition.

The result is then established as follows, by imagining a standard reduction sequence for some given coherence term. Here we refer to composite relations by concatenation; so for example, $t \overset{\mathrm{B}\,\mathrm{A}}{\rightsquigarrow\rightsquigarrow} t'$ just when there exists some t'' with $t \overset{\mathrm{B}}{\rightsquigarrow} t'' \overset{\mathrm{A}}{\rightsquigarrow} t'$.

- Standard A-reductions have the highest priority, so these will be performed first.

- If the above step terminates, standard B-reductions have the second-highest priority, so we now perform these. Since $\overset{\mathrm{B}\,\mathrm{A}}{\rightsquigarrow\rightsquigarrow}$ is empty, this will not trigger any additional standard A-reductions.

- If the above step terminates, standard C-reductions have the third-highest priority, so we now perform these. Since $\overset{\mathrm{C}\,\mathrm{A}}{\rightsquigarrow\rightsquigarrow}$ and $\overset{\mathrm{C}\,\mathrm{B}}{\rightsquigarrow\rightsquigarrow}$ are both empty, these standard C-reductions will not trigger further standard A- or B-reductions.

- If the above step terminates, standard D-reductions have the fourth-highest priority. If we can perform a D-reduction, the result will be in normal form, since $\overset{\mathrm{D}}{\rightsquigarrow}\rightsquigarrow$ is empty.

- If we cannot perform a D-reduction, we consider applying a standard E-reduction, as the standard reduction with fifth-highest priority. If the standard E-reduction cannot be applied, then the term is in normal form, as the standard E-reduction is the last reduction in the list.

- If the standard E-reduction was successfully applied, it cannot be followed by a standard B-, C-, D-, or E- reduction, since $\overset{\mathrm{E}\,\mathrm{B}}{\rightsquigarrow\rightsquigarrow}$, $\overset{\mathrm{E}\,\mathrm{C}}{\rightsquigarrow\rightsquigarrow}$, $\overset{\mathrm{E}\,\mathrm{D}}{\rightsquigarrow\rightsquigarrow}$ and $\overset{\mathrm{E}\,\mathrm{E}}{\rightsquigarrow\rightsquigarrow}$ are all shown above to be empty. The only remaining possibility is for the standard E-reduction to be followed by some sequence of standard A-reductions. These A-reductions cannot themselves be followed by a standard B-, C-, D- or E-reduction, since we show above that $\overset{\mathrm{E}}{\rightsquigarrow}\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}}\overset{\mathrm{B}}{\rightsquigarrow}$, $\overset{\mathrm{E}}{\rightsquigarrow}\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}}\overset{\mathrm{C}}{\rightsquigarrow}$, $\overset{\mathrm{E}}{\rightsquigarrow}\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}}\overset{\mathrm{D}}{\rightsquigarrow}$ and $\overset{\mathrm{E}}{\rightsquigarrow}\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}}\overset{\mathrm{E}}{\rightsquigarrow}$ are all empty.

This completes the proof. $\square$

With this in hand, we can show termination.

Proposition 31. *Standard reduction is terminating on valid types, terms and substitutions.*

Proof. Standard reduction on types and substitutions is given in terms of standard reduction of a finite family of terms, so we need only check that standard reduction of terms has no infinite sequences. The variable case is trivial, so we consider reduction of some coherence term $t \equiv \mathsf{coh}\,(\Gamma : T)[\sigma]$.

We proceed by simultaneous induction, on the dimension of t, and on subterms of t. Since a subterm of a valid term can never have a higher dimension, this is well-defined. The dimension is not defined for variables, but since variables are in normal form, this does not affect the argument. We also note that any subterm of a valid term is itself valid.

Thanks to Proposition 30, we know that the standard reduction of a given term can be separated into finitely many distinct phases of standard A-, B-, C-, D- and E-reduction, with each phase involving standard reductions of a single fixed type. So we need only show that each standard reduction phase will terminate.

- *Standard A-reduction.* For the first phase of A-reductions, we can use an induction on subterms, since a substitution is of finite length, a given term will have only finitely many standard A-reductions. For the second phase of A-reductions, we have that these reductions followed an E-reduction, and so all the terms in the substitution have a lower dimension than the original term, and so we can induct on dimension and apply the same reasoning as before.
- *Standard B-reduction.* Since the context has a finite number of variables, a given term will have only finitely many standard B-reductions.
- *Standard C-reduction.* Since the cell part of a valid term is formed from terms of strictly lower dimension, and is itself valid in the pasting context of the head coherence, it follows by induction on dimension that a term will have only finitely many standard C-reductions.

Standard D- and E-reductions are single-step operations, so no termination argument is necessary for those. □

Having now established that standard reduction has no infinite sequences, it is clear every term t has a unique normal form, $N(t)$, to which it reduces.

3.4 Technical results on reduction

Here we collect further results on reduction, mostly of a technical nature, which will be used in the next subsection for our main proof.

Lemma 32. *Term substitution is compatible with substitution reduction:*

$$\sigma \rightsquigarrow \sigma' \quad \Rightarrow \quad u[\![\sigma]\!] \rightsquigarrow_{\mathsf{rt}} u[\![\sigma']\!]$$

Proof. We induct on the structure of u. If u is a variable, then either $u[\![\sigma]\!] \equiv \sigma'[\![\tau]\!]$ or $u[\![\sigma]\!] \rightsquigarrow u[\![\sigma']\!]$; in either case, we have $u[\![\sigma]\!] \rightsquigarrow_{\mathsf{r}} u[\![\sigma']\!]$ as required.

Otherwise, we have $u \equiv \mathsf{coh}\,(\Gamma : U)[\rho]$, and we argue as follows:

$$\begin{aligned}
u[\sigma] &\equiv \mathsf{coh}\,(\Gamma : U)[\rho \circ \sigma] \\
&\equiv \mathsf{coh}\,(\Gamma : U)[\rho_1[\![\sigma]\!], \rho_2[\![\sigma]\!], \ldots, \rho_n[\![\sigma]\!]] \\
&\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} \mathsf{coh}\,(\Gamma : U)[\rho_1[\![\sigma']\!], \rho_2[\![\sigma]\!], \ldots, \rho_n[\![\sigma]\!]] \\
&\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} \mathsf{coh}\,(\Gamma : U)[\rho_1[\![\sigma']\!], \rho_2[\![\sigma']\!], \ldots, \rho_n[\![\sigma]\!]] \\
&\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} \cdots \\
&\overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rt}} \mathsf{coh}\,(\Gamma : U)[\rho_1[\![\sigma']\!], \rho_2[\![\sigma']\!], \ldots, \rho_n[\![\sigma']\!]] \\
&\equiv \mathsf{coh}\,(\Gamma : U)[\rho \circ \sigma'] \\
&\equiv u[\![\sigma']\!]
\end{aligned}$$

Hence $u[\![\sigma]\!] \rightsquigarrow_{\mathsf{rt}} u[\![\sigma']\!]$ as required. □

Lemma 33. *Term substitution is compatible with term reduction:*

$$u \rightsquigarrow u' \quad \Rightarrow \quad u[\![\sigma]\!] \rightsquigarrow u'[\![\sigma]\!]$$

Proof. If u is a variable, no reduction is possible, in contradiction with hypothesis. We therefore assume $u \equiv \mathsf{coh}\,(\Gamma : T)[\mu]$ is a coherence term, writing $\mu = [\mu_1, \ldots, \mu_n]$, and proceed by case analysis on the structure of the reduction $u \rightsquigarrow u'$, and by induction on subterms of u.

If $u \overset{\mathrm{A}}{\rightsquigarrow} u'$ via some $m_i \rightsquigarrow m_i'$, then we argue as follows:

$$\begin{aligned} u[\![\sigma]\!] &\equiv \mathsf{coh}\,(\Gamma : U)[\mu \circ \sigma] \\ &\equiv \mathsf{coh}\,(\Gamma : U)[\mu_1[\![\sigma]\!], \ldots, \mu_i[\![\sigma]\!], \ldots, \mu_k[\![\sigma]\!]] \\ &\overset{\mathrm{A}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : U)[\mu_1[\![\sigma]\!], \ldots, \mu_i'[\![\sigma]\!], \ldots, \mu_k[\![\sigma]\!]] \\ &\equiv u[\![\sigma']\!] \end{aligned}$$

Alternatively, we suppose $u \overset{\mathrm{B}}{\rightsquigarrow} u'$ is the B-reduction $\mathsf{coh}\,(\Gamma : U)[\mu] \overset{\mathrm{B}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma /\!\!/ x : U[\![\pi_x]\!])[\mu /\!\!/ x]$, eliminating some locally-maximal variable x_i of Γ for which $x_i[\![\mu]\!] \equiv \mu_i$ is an identity. Then $\mu_i[\![\sigma]\!]$ is also an identity, and hence using Lemma 9 we have:

$$\begin{aligned} u[\![\sigma]\!] &\equiv \mathsf{coh}\,(\Gamma : U)[\mu \circ \sigma] \\ &\equiv \mathsf{coh}\,(\Gamma : U)[\mu \circ \sigma] \\ &\overset{\mathrm{B}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma /\!\!/ x : U[\![\pi_x]\!])[(\mu \circ \sigma) /\!\!/ x] \\ &\equiv \mathsf{coh}\,(\Gamma /\!\!/ x : U[\![\pi_x]\!])[(\mu /\!\!/ x) \circ \sigma] \\ &\equiv u'[\![\sigma]\!] \end{aligned}$$

If $u \overset{\mathrm{C}}{\rightsquigarrow} u'$ via some $T \overset{\mathrm{C}}{\rightsquigarrow} T'$, then we argue as follows:

$$u[\![\sigma]\!] \equiv \mathsf{coh}\,(\Gamma : T)[\mu \circ \sigma] \overset{\mathrm{C}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma : T')[\mu \circ \sigma] \equiv u'[\![\sigma]\!]$$

If $u \overset{\mathrm{D}}{\rightsquigarrow} u'$ as $\mathsf{coh}\,(\mathrm{D}^n : S^{n-1})[\ldots, u'] \overset{\mathrm{D}}{\rightsquigarrow} u'$, then we argue as follows:

$$u[\![\sigma]\!] \equiv \mathsf{coh}\,(\mathrm{D}^n : S^{n-1})[\ldots, u'[\![\sigma]\!]] \overset{\mathrm{D}}{\rightsquigarrow} u'[\![\sigma]\!]$$

Finally, if $u \overset{\mathrm{E}}{\rightsquigarrow} u'$ as $\mathsf{coh}\,(\Gamma : u \to_U u)[\mu] \overset{\mathrm{E}}{\rightsquigarrow} \mathbb{1}_{\mathsf{dim} U+1}[\![\{U, u\} \circ \mu]\!]$, then we argue as follows, using Lemma 2:

$$\begin{aligned} u[\![\sigma]\!] &\equiv \mathsf{coh}\,(\Gamma : u \to_U u)[\mu \circ \sigma] \\ &\overset{\mathrm{E}}{\rightsquigarrow} \mathbb{1}_{\mathsf{dim} U+1}[\![\{U, u\} \circ \mu \circ \sigma]\!] \\ &\equiv \mathbb{1}_{\mathsf{dim} U+1}[\![\{U, u\} \circ \mu]\!][\![\sigma]\!] \\ &\equiv u'[\![\sigma]\!] \end{aligned}$$

This completes the proof. $\square$

Lemma 34 (Identities reduce to identities)**.** *If u is an identity, and $u \rightsquigarrow u'$, then u' is an identity.*

Proof. We recognize an identity term by looking at the head. We prove the result by case analysis on the reduction $u \rightsquigarrow u'$. If $u \overset{\mathrm{A}}{\rightsquigarrow} u'$ the result is immediate, since A-reductions do not change the head. If $u \overset{\mathrm{B}}{\rightsquigarrow} u'$ we have a contradiction, since identity terms cannot be B-redexes by definition. If $u \overset{\mathrm{C}}{\rightsquigarrow} u'$ we again have a contradiction, since a C-reduction acts on the head of the term, but the head of an identity term is in normal form. If $u \overset{\mathrm{D}}{\rightsquigarrow} u'$ we again have a contradiction, since identity terms have the wrong form to be D-redexes. If $u \overset{\mathrm{E}}{\rightsquigarrow} u'$ we again have a contradiction, since identity terms cannot be E-redexes by definition. □

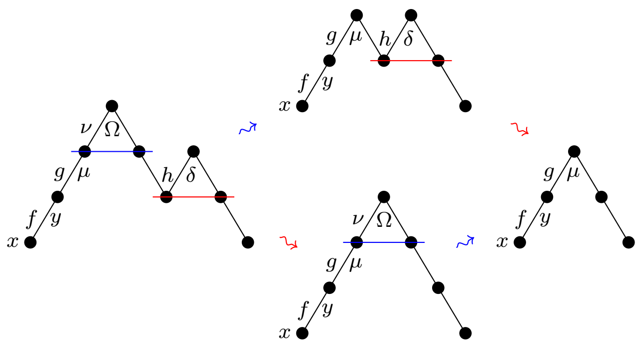
Lemma 35. *If $\sigma \rightsquigarrow \tilde{\sigma}$, then $\sigma /\!/ x \rightsquigarrow_{\mathsf{r}} \tilde{\sigma} /\!/ x$.*

Proof. This is immediate, since $\sigma /\!/ x$ is a sublist of σ. If the first reducible argument of σ is not in the sublist $\sigma /\!/ x$, then $\sigma /\!/ x \equiv \tilde{\sigma} /\!/ x$. Otherwise, it will still be the first reducible argument of the sublist, and $\sigma /\!/ x \rightsquigarrow \tilde{\sigma} /\!/ x$. □

Lemma 36. *Given a context Γ and distinct locally-maximal variables x, y, and a substitution $\Delta \vdash \sigma : \Gamma$ with $x[\![\sigma]\!], y[\![\sigma]\!]$ both identities, the following contexts and substitutions are identical:*

$$(\Gamma /\!/ x) /\!/ y \equiv (\Gamma /\!/ y) /\!/ x \qquad \pi_x \circ \pi_y \equiv \pi_y \circ \pi_x \qquad (\sigma /\!/ x) /\!/ y \equiv (\sigma /\!/ y) /\!/ x$$

Proof. The first statement is clear from example, in this case showing $(\Gamma /\!/ \Omega) /\!/ \delta \equiv (\Gamma /\!/ \delta) /\!/ \Omega$, since the excision operations are independent:



The other claims follows similarly. □

Lemma 37. *For a valid substitution $\Delta \vdash \sigma : \mathrm{D}^n$, we have for all $i \leq n$:*

$$d_i[\![\sigma]\!] = \mathsf{src}^{n-i}(d_n[\![\sigma]\!]) \qquad d'_i[\![\sigma]\!] = \mathsf{tgt}^{n-i}(d_n[\![\sigma]\!])$$

Proof. Extending the substitution construction rule along definitional equality we must have $\mathsf{src}(d_n[\![\sigma]\!]) = \mathsf{src}(d_n)[\![\sigma]\!]$, and similarly for tgt. The result follows. □

3.5 Standard reduction generates definitional equality

We will show that the symmetric, transitive, reflexive closure of standard reduction generates definitional equality. Since we have already shown that standard reduction is a terminating reduction strategy, this gives an algorithm to determine whether two given terms are definitionally equal, by computing their standard normal forms and checking if they are syntactically equal (up to α-equivalence.)

Lemma 38. *If $a \boldsymbol{\rightsquigarrow}_{\mathsf{rts}} b$, $\exists\, c$ with $a \boldsymbol{\rightsquigarrow}_{\mathsf{rt}} c$ and $b \boldsymbol{\rightsquigarrow}_{\mathsf{rt}} c$.*

Proof. This is immediate, since $\boldsymbol{\rightsquigarrow}$ is a reduction strategy. □

The next proposition tells us that standard reduction agrees with equality.

Proposition 39. *For all terms s, such that s is valid in some context Θ, we have the following:*

(i) If $s \rightsquigarrow t$, we can find terms a, b which admit a reduction $a \rightsquigarrow_{\mathsf{r}} b$ and standard reductions $s \boldsymbol{\rightsquigarrow}_{\mathsf{t}} a$, $t \boldsymbol{\rightsquigarrow}_{\mathsf{rts}} b$, illustrated as follows:

$$\begin{tikzcd} s \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow, "\mathsf{t}"'] & t \arrow[d, rightsquigarrow, "\mathsf{rts}"] \\ a \arrow[r, rightsquigarrow, "\mathsf{r}"'] & b \end{tikzcd}$$

(ii) If $\Theta \vdash s = t$, then $s \boldsymbol{\rightsquigarrow}_{\mathsf{rts}} t$.

Proof. Since by Proposition 31 every term reaches normal form after a finite number of standard reductions, and since by Lemma 20 reduction generates equality, it is clear that statement (i) implies statement (ii). We therefore focus here on the proof of (i).

We can neglect the case of s being a variable, since variables do not reduce. It follows that s is a coherence term, and since the dimension of a valid coherence term is well-defined, we will make use of that throughout. The proof of (i) is by simultaneous induction on the dimension of s, and on subterms of s. Since no subterm of s has a greater dimension than s itself, this is consistent. We also note that if we have property (i) for all terms of dimension less than n, then we can derive property (ii) for all terms of dimension less than n, and so we may assume (ii) holds on terms of lower dimension by induction on dimension.

One possibility, which arises several times in the case analysis below, is that the reduction $s \rightsquigarrow t$ is itself standard (that is, $t \equiv \tilde{s}$.) We can handle this case once-and-for-all as follows:

$$\begin{tikzcd} s \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & t \arrow[d, equal] \\ \tilde{s} \arrow[r, equal] & \tilde{s} \end{tikzcd} \tag{1}$$

We refer to this argument below where it is needed.

We now begin the main proof of property (i), by case analysis on the reduction $s \rightsquigarrow t$.

• *First case $s \overset{\mathrm{A}}{\rightsquigarrow} t$.* We suppose $s \equiv \mathsf{coh}\,(\Gamma : T)[\sigma]$, and that $s \overset{\mathrm{A}}{\rightsquigarrow} t$ by reducing an argument of σ via $s_i \rightsquigarrow s_i'$. Then s must be a standard A-redex, because if some argument is not in normal form, there must exist a leftmost argument not in normal form; so we have $s \overset{\mathrm{A}}{\boldsymbol{\rightsquigarrow}} \tilde{s}$.

Suppose $s \overset{\mathrm{A}}{\rightsquigarrow} t$ and $s \overset{\mathrm{A}}{\boldsymbol{\rightsquigarrow}} \tilde{s}$ act by reducing the same argument of s_i of σ, via $s_i \rightsquigarrow s_i'$ and $s_i \boldsymbol{\rightsquigarrow} \tilde{s}_i$ respectively. If $s_i' \equiv \tilde{s}_i$ then we are done by the argument above expression (1) above. Otherwise the result holds by induction on the subterm s_i, as follows:

$$\begin{tikzcd} \mathsf{coh}\,(\Gamma : T)[\ldots, s_i, \ldots] \arrow[r, rightsquigarrow, "\mathrm{A}"] \arrow[d, rightsquigarrow, "\mathrm{A}"'] & \mathsf{coh}\,(\Gamma : T)[\ldots, s_i', \ldots] \arrow[d, rightsquigarrow, "\mathsf{rts}"', "\mathrm{A}"] \\ \mathsf{coh}\,(\Gamma : T)[\ldots, \tilde{s}_i, \ldots] \arrow[r, rightsquigarrow, "\mathrm{A}"'] & \mathsf{coh}\,(\Gamma : T)[\ldots, q, \ldots] \end{tikzcd}$$

Alternatively, suppose they act by reducing different arguments of σ. Then the redexes are independent, and we argue as follows:

$$
\begin{tikzcd}
\mathsf{coh}\,(\Gamma : T)[\ldots, s_i, \ldots, s_j, \ldots] \arrow[r, rightsquigarrow, "\mathrm{A}"] \arrow[d, rightsquigarrow, "\mathrm{A}"'] & \mathsf{coh}\,(\Gamma : T)[\ldots, s_i, \ldots, s'_j, \ldots] \arrow[d, rightsquigarrow, "\mathrm{A}"] \\
\mathsf{coh}\,(\Gamma : T)[\ldots, \tilde{s}_i, \ldots, s_j, \ldots] \arrow[r, rightsquigarrow, "\mathrm{A}"'] & \mathsf{coh}\,(\Gamma : T)[\ldots, \tilde{s}_i, \ldots, s'_j, \ldots]
\end{tikzcd}
$$

- *Second case* $s \overset{\mathrm{B}}{\rightsquigarrow} t$. For this case, we suppose $s \overset{\mathrm{B}}{\rightsquigarrow} t$ as follows:

$$s \equiv \mathsf{coh}\,(\Gamma : T)[\sigma] \overset{\mathrm{B}}{\rightsquigarrow} \mathsf{coh}\,(\Gamma /\!/ x : T[\![\pi_x]\!])[\sigma /\!/ x] \equiv t$$

We proceed by case analysis on the standard reduction $s \rightsquigarrow \tilde{s}$.

Standard A-reduction. In this case, we argue as follows:

$$
\begin{tikzcd}
\mathsf{coh}\,(\Gamma : T)[\sigma] \arrow[r, rightsquigarrow, "\mathrm{B}"] \arrow[d, rightsquigarrow, "\mathrm{A}"'] & \mathsf{coh}\,(\Gamma /\!/ x : T[\![\pi_x]\!])[\sigma /\!/ x] \arrow[d, rightsquigarrow, "\mathrm{A}"] \\
\mathsf{coh}\,(\Gamma : T)[\tilde{\sigma}] \arrow[r, rightsquigarrow, "\mathrm{B}"'] & \mathsf{coh}\,(\Gamma /\!/ x : T[\![\pi_x]\!])[\tilde{\sigma} /\!/ x]
\end{tikzcd}
$$

Since the upper B-reduction is valid, we know that $x[\![\sigma]\!]$ is an identity; then by Lemma 34 we also have that $x[\![\tilde{\sigma}]\!]$ is an identity, and so the lower B-reduction is valid. Validity of the standard A-reduction on the right of the square follows from Lemma 35.

Standard B-reduction. We suppose s is a standard B-redex with respect to some locally-maximal variable y. If $x \equiv y$, then the reductions are the same, and this case is handled by the general argument above expression (1). Otherwise, we argue as follows, using Lemma 36:

$$
\begin{tikzcd}
\mathsf{coh}\,(\Gamma : T)[\sigma] \arrow[r, rightsquigarrow, "\mathrm{B}"] \arrow[dd, rightsquigarrow, "\mathrm{B}"'] & \mathsf{coh}\,(\Gamma /\!/ x : T[\![\pi_x]\!])[\sigma /\!/ x] \arrow[d, rightsquigarrow, "\mathrm{B}"] \\
& \mathsf{coh}\,\big((\Gamma /\!/ x) /\!/ y : T[\![\pi_x]\!][\![\pi_y]\!]\big)[(\sigma /\!/ x) /\!/ y] \arrow[d, equal] \\
\mathsf{coh}\,(\Gamma /\!/ y : T[\![\pi_y]\!])[\sigma /\!/ y] \arrow[r, rightsquigarrow, "\mathrm{B}"'] & \mathsf{coh}\,\big((\Gamma /\!/ y) /\!/ x : T[\![\pi_y]\!][\![\pi_x]\!]\big)[(\sigma /\!/ y) /\!/ x]
\end{tikzcd}
$$

- *Third case* $s \overset{\mathrm{C}}{\rightsquigarrow} t$. Suppose $s \equiv \mathsf{coh}\,(\Gamma : T)[\sigma]$, and $s \overset{\mathrm{C}}{\rightsquigarrow} t$ acts via some type reduction $T \rightsquigarrow U$. Then we argue by case analysis on the reduction $s \rightsquigarrow \tilde{s}$.

Standard A-reduction. In this case we have the following:

$$
\begin{tikzcd}
\mathsf{coh}\,(\Gamma : T)[\sigma] \arrow[r, rightsquigarrow, "\mathrm{C}"] \arrow[d, rightsquigarrow, "\mathrm{A}"'] & \mathsf{coh}\,(\Gamma : U)[\sigma] \arrow[d, rightsquigarrow, "\mathrm{A}"] \\
\mathsf{coh}\,(\Gamma : T)[\tilde{\sigma}] \arrow[r, rightsquigarrow, "\mathrm{C}"'] & \mathsf{coh}\,(\Gamma : U)[\tilde{\sigma}]
\end{tikzcd}
$$

Standard B-reduction. If s is a standard B-redex, we have the following, employing Lemma 33:

$$\begin{tikzcd}
\mathsf{coh}\,(\Gamma : T)[\sigma] \arrow[r, squiggly, "\mathrm{C}"] \arrow[d, squiggly, "\mathrm{B}"'] & \mathsf{coh}\,(\Gamma : U)[\sigma] \arrow[d, squiggly, "\mathrm{B}"] \\
\mathsf{coh}\,(\Gamma /\!\!/ x : T[\![\pi_x]\!])[\sigma /\!\!/ x] \arrow[r, squiggly, "\mathrm{C}"'] & \mathsf{coh}\,(\Gamma /\!\!/ x : U[\![\pi_x]\!])[\sigma /\!\!/ x]
\end{tikzcd}$$

Standard C-reduction. Suppose s is a standard C-redex via a standard type reduction $T \boldsymbol{\rightsquigarrow} \tilde{T}$. If $\tilde{T} \equiv U$, we are done by the argument above expression (1). Otherwise, since $T \rightsquigarrow U$, by induction on subterms we know that $U \boldsymbol{\rightsquigarrow}_{\mathsf{rts}} T$, and we argue as follows:

$$\begin{tikzcd}
\mathsf{coh}\,(\Gamma : T)[\sigma] \arrow[r, squiggly, "\mathrm{C}"] \arrow[dd, squiggly, "\mathrm{C}"', "\mathsf{t}"] & \mathsf{coh}\,(\Gamma : U)[\sigma] \arrow[d, squiggly, "\mathsf{rts}"', "\mathrm{C}"] \\
 & \mathsf{coh}\,(\Gamma : T)[\tau] \arrow[d, squiggly, "\mathrm{C}"] \\
\mathsf{coh}\,(\Gamma : \tilde{T})[\tau] \arrow[r, equal] & \mathsf{coh}\,(\Gamma : \tilde{T})[\tau]
\end{tikzcd}$$

- *Fourth case $s \overset{\mathrm{D}}{\rightsquigarrow} t$.* We suppose $s \overset{\mathrm{D}}{\rightsquigarrow} t$ as follows:

$$s \equiv \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, t] \overset{\mathrm{D}}{\rightsquigarrow} t$$

Then we consider the standard reduction for s.

Standard A-reduction. Supposing $t \boldsymbol{\rightsquigarrow} \tilde{t}$ is the leftmost reducible argument of s, then we have the following:

$$\begin{tikzcd}
\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, t] \arrow[r, squiggly, "\mathrm{D}"] \arrow[d, squiggly, "\mathrm{A}"'] & t \arrow[d, squiggly] \\
\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, \tilde{t}] \arrow[r, squiggly, "\mathrm{D}"'] & \tilde{t}
\end{tikzcd}$$

Otherwise, let p be the leftmost reducible argument of s. Then we argue as follows:

$$\begin{tikzcd}
\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, p, \ldots, t] \arrow[r, squiggly, "\mathrm{D}"] \arrow[d, squiggly, "\mathrm{A}"'] & t \arrow[d, equal] \\
\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, \tilde{p}, \ldots, t] \arrow[r, squiggly, "\mathrm{D}"'] & t
\end{tikzcd}$$

Standard B-reduction. We must have $t \equiv \mathbb{1}_n[\![\ldots, q_2, q_1]\!]$, and hence $s \equiv \mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, p_2, p_1, p_1', \mathbb{1}_n[\![\ldots, q_2, q_1]\!]]$. Since s is valid we deduce $q_1 = p_1 = p_1'$ and $q_i = p_i$. It follows by induction on dimension that $q_i \boldsymbol{\rightsquigarrow}_{\mathsf{rts}} p_i$. We put this together as follows:

$$\begin{tikzcd}
\mathsf{coh}\,(\mathrm{D}^{n+1} : S^n)[\ldots, p_2, p_1, p_1', \mathbb{1}_n[\![\ldots, q_2, q_1]\!]] \arrow[r, squiggly, "\mathrm{D}"] \arrow[d, squiggly, "\mathrm{B}"'] & \mathbb{1}_n[\![\ldots, q_2, q_1]\!] \arrow[d, squiggly, "\mathsf{rts}"', "\mathrm{A}"] \\
\mathsf{coh}\,(\mathrm{D}^n : d_n \to_{S^{n-1}} d_n)[\ldots, p_2, p_1] \arrow[r, equal] & \mathbb{1}_n[\![\ldots, p_2, p_1]\!]
\end{tikzcd}$$

Standard C-reduction. The term s cannot be a C-redex, since the type S^n is in normal form, being constructed entirely from variables.

Standard D-reduction. In this case, the result follows from the argument around around expression (1) above.

- *Fifth case* $s \overset{\mathrm{E}}{\rightsquigarrow} t$. We suppose $s \overset{\mathrm{E}}{\rightsquigarrow} t$ as follows, for $n = \mathsf{dim}\, T + 1$:

$$s \equiv \mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] \rightsquigarrow \mathbb{1}_n[\![\{T, u\} \circ \sigma]\!] \equiv t$$

We now consider the structure of the standard reduction $s \rightsquigarrow \tilde{s}$.

Standard A-reduction. If $\sigma \rightsquigarrow \tilde{\sigma}$ then we get by Lemma 8 that $\{T, u\} \circ \sigma = \{T, u\} \circ \tilde{\sigma}$. We conclude by induction on dimension that $\mathbb{1}_n[\![\{T, u\} \circ \sigma]\!] \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rts}} \mathbb{1}_n[\![\{T, u\} \circ \tilde{\sigma}]\!]$. Altogether, we have the following as required:

$$\begin{tikzcd}
\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] \arrow[r, squiggly, "\mathrm{E}"] \arrow[d, squiggly, "\mathrm{A}"'] & \mathbb{1}_n[\![\{T, u\} \circ \sigma]\!] \arrow[d, squiggly, "\mathsf{rts}"', "\mathrm{A}"] \\
\mathsf{coh}\,(\Gamma : u \to_T u)[\tilde{\sigma}] \arrow[r, squiggly, "\mathrm{E}"'] & \mathbb{1}_n[\![\{T, u\} \circ \tilde{\sigma}]\!]
\end{tikzcd}$$

Standard B-reduction. In this case we have the following:

$$\begin{tikzcd}
\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] \arrow[r, squiggly, "\mathrm{E}"] \arrow[d, squiggly, "\mathrm{B}"'] & \mathbb{1}_n[\{T, u\} \circ \sigma] \arrow[d, squiggly, "\mathsf{rts}"', "\mathrm{A}"] \\
\mathsf{coh}\,(\Gamma /\!/ x : (u \to_T u)[\pi_x])[\sigma /\!/ x] \arrow[r, squiggly, "\mathrm{E}"'] & \mathbb{1}_n[\{T[\pi_x], u[\pi_x]\} \circ \sigma /\!/ x]
\end{tikzcd}$$

We obtain the right-hand standard A-reduction as follows. From Lemma 11 we know $\sigma = \pi_x \circ (\sigma /\!/ x)$, and hence:

$$\{T[\![\pi_x]\!], u[\![\pi_x]\!]\} \circ \sigma /\!/ x \equiv \{T, u\} \circ \pi_x \circ \sigma /\!/ x = \{T, u\} \circ \sigma$$

By induction on dimension we conclude $\{T, u\} \circ \sigma \rightsquigarrow_{\mathsf{rts}} \{T[\![\pi_x]\!], u[\![\pi_x]\!]\} \circ \sigma /\!/ x$.

Standard C-reduction. Since $\{T, u\} = \{N(T), N(u)\}$ and so $\{T, u\} \circ \sigma = \{N(T), N(u)\} \circ \sigma$, it follows by induction on dimension that $\{T, u\} \circ \sigma \rightsquigarrow_{\mathsf{rts}} \{N(T), N(u)\} \circ \sigma$, and hence that $\mathbb{1}_n[\![\{T, u\} \circ \sigma]\!] \overset{\mathrm{A}}{\rightsquigarrow}_{\mathsf{rts}} \mathbb{1}_n[\![\{N(T), N(u)\} \circ \sigma]\!]$. We then have the following:

$$\begin{tikzcd}
\mathsf{coh}\,(\Gamma : u \to_T u)[\sigma] \arrow[r, squiggly, "\mathrm{E}"] \arrow[d, squiggly, "\mathrm{C}"', "\mathsf{t}"] & \mathbb{1}_n[\![\{T, u\} \circ \sigma]\!] \arrow[d, squiggly, "\mathsf{rts}"', "\mathrm{A}"] \\
\mathsf{coh}\,(\Gamma : N(u) \to_{N(T)} N(u))[\sigma] \arrow[r, squiggly, "\mathrm{E}"'] & \mathbb{1}_n[\![\{N(T), N(u)\} \circ \sigma]\!]
\end{tikzcd}$$

Standard D-reduction. The term s cannot be a D-redex, as the term s has an incompatible structure.

Standard E-reduction. In this case we are done, with the result following from the argument around around expression (1).

This completes the argument. ☐

Finally, we can use these results to show that the equality relation is decidable.

Theorem 40. *We have $s = t$ if and only if $N(s) \equiv N(t)$.*

Proof. If $N(s) \equiv N(t)$ then $s \rightsquigarrow_{\mathsf{rts}} t$ and so $s = t$. Conversely, if $s = t$, then by Proposition 39, we have that $s \rightsquigarrow_{\mathsf{rts}} t$, and so by Lemma 38 we have $s \rightsquigarrow_{\mathsf{rt}} u$ and $t \rightsquigarrow_{\mathsf{rt}} u$ and so $N(s) \equiv N(u) \equiv N(t)$. ☐

4 Examples and Implementation

In this section, we investigate some examples, to see the type theory in action. As we can decide equality by comparing normal forms, we are able to provide an algorithm for type-checking $\mathsf{Catt}_{\mathsf{su}}$ terms. We have implemented type theory in OCaml, and made it available here:

`http://github.com/ericfinster/catt.io/tree/v0.1`

The following examples demonstrate the use of this implementation, and the code for all of them can be found in the examples folder within the repository.

Example 41 (Left unitor).
`examples/example_left_unitor.catt`
The left unitor on $f : x \to y$ is defined by the following Catt term, of type $\mathsf{comp}_2[\![\mathbb{1}_0[\![x]\!], f]\!] \to f$:

$$\mathsf{coh}\,((x:\star)(y:\star)(f : x \to_\star y) : \mathsf{comp}_2[\![x, x, \mathbb{1}_0[\![x]\!], y, f]\!] \to f)[x, y, f]$$

In $\mathsf{Catt}_{\mathsf{su}}$ this reduces to the identity on f, demonstrating how higher coherence data can trivialise in the theory.

Example 42 (Associator on a unit).
`examples/example_associator_unit.catt`
For $f : x \to y$ and $g : y \to z$, we have the following defin-itional equality in $\mathsf{Catt}_{\mathsf{su}}$:

$$(f \cdot \mathrm{id}_y) \cdot g = f \cdot (\mathrm{id}_y \cdot g)$$

We may therefore anticipate the associator $\alpha_{f,\mathrm{id}_y,g}$ to trivialise, and indeed this is the case. We illustrate this with the following standard reduction sequence, where for brevity we leave some substitution arguments implicit:

$$\begin{aligned}
&\mathsf{assoc}[\![f, \mathbb{1}_0[\![y]\!], g]\!] \\
\equiv\ &\mathsf{coh}\,((x:\star)(y:\star)(a : x \to y)(z:\star)(b : y \to z)(w:\star)(c : z \to w) : \\
&\quad \mathsf{comp}_2[\![\mathsf{comp}_2[\![a, b]\!], c]\!] \to \mathsf{comp}_2[\![a, \mathsf{comp}_2[\![b, c]\!]]\!])[\![f, \mathbb{1}_0[\![y]\!], g]\!] \\
\overset{\mathrm{B}}{\rightsquigarrow}\ &\mathsf{coh}\,((x:\star)(y:\star)(a : x \to y)(w:\star)(c : y \to w) : \\
&\quad \mathsf{comp}_2[\![\mathsf{comp}_2[\![a, \mathbb{1}_0[\![y]\!]]\!], c]\!] \to \mathsf{comp}_2[\![a, \mathsf{comp}_2[\![\mathbb{1}_0[\![y]\!], c]\!]]\!])[\![f, g]\!] \\
\overset{\mathrm{C}}{\rightsquigarrow}_{\mathsf{t}}\ &\mathsf{coh}\,((x:\star)(y:\star)(a : x \to y)(w:\star)(c : y \to w) : \\
&\quad \mathsf{comp}_2[\![\mathsf{comp}_1[\![a]\!], c]\!] \to \mathsf{comp}_2[\![a, \mathsf{comp}_1[\![c]\!]]\!])[\![f, g]\!] \\
\overset{\mathrm{C}}{\rightsquigarrow}_{\mathsf{t}}\ &\mathsf{coh}\,((x:\star)(y:\star)(a : x \to y)(w:\star)(c : y \to w) : \\
&\quad \mathsf{comp}_2[\![a, c]\!] \to \mathsf{comp}_2[\![a, c]\!])[\![f, g]\!] \\
\overset{\mathrm{E}}{\rightsquigarrow}\ &\mathbb{1}_1[\![\{(x \to z), \mathsf{comp}_2[\![f, g]\!]\}]\!]
\end{aligned}$$

This reduction sequence uses all three generators PRUNE, DISC and ENDO, with disc removal being used for the second C-reduction.

Example 43 (Eckmann-Hilton move).
`examples/example_eh_catt.catt`
`examples/example_eh_cattsu.catt`

The *Eckmann-Hilton move* of higher category theory is an algebraic phenomenon that arises in the following context:

$$(x : \star)(a, b : \mathbb{1}_0[\![x]\!] \to \mathbb{1}_0[\![x]\!])$$

The claim is that one can construct a composite 3-cell of the following type:

$$\mathsf{EH}_3[\![a, b]\!] : \mathsf{comp}_2[\![a, b]\!] \to \mathsf{comp}_2[\![b, a]\!]$$

The geometrical intuition is that a, b "braid" around each other in the plane.

This Eckmann-Hilton 3-cell has been previously construc-ted in Catt, and we link to its formalization above. It is a geometrically complex proof, involving substantial use of the weak unit structure, requiring in total 1574 coherence constructors in the proof term.

Given the substantial use of the unit structure, we would anticipate the proof would simplify considerably in $\mathsf{Catt}_{\mathsf{su}}$ when normalized. This is indeed what we observe, with the $\mathsf{Catt}_{\mathsf{su}}$ normal form involving only 27 coherence constructors in the proof term, a 50-fold reduction in complexity.

Links to both the Catt and $\mathsf{Catt}_{\mathsf{su}}$ formalizations are provided at the top of this example. The $\mathsf{Catt}_{\mathsf{su}}$ normal form is so short that we are able to provide it explicitly as Appendix A.

Example 44 (Syllepsis)**.**
`examples/example_syllepsis.catt`
The Eckmann-Hilton move "braids" the generators a, b past each other. After some consideration it becomes clear that there should be two homotopically distinct ways to achieve this, which could be called the "braid" and "inverse braid", and we would not expect these proofs to be equivalent.

However, if we increase the level of degeneracy in the types, these moves in fact become equivalent. To show this, we must instead use the following context:

$$(x : \star)(\alpha, \beta : \mathbb{1}_1[\![\mathbb{1}_0[\![x]\!]]\!] \to \mathbb{1}_1[\![\mathbb{1}_0[\![x]\!]]\!])$$

Here α, β are 3-dimensional generators. We can adapt our definition of the EH cell to this higher dimension, yielding the following 4-cell:

$$\mathsf{EH}_4[\![\alpha, \beta]\!] : \mathsf{comp}_3[\![\alpha, \beta]\!] \to \mathsf{comp}_3[\![\beta, \alpha]\!]$$

The syllepsis is then a 5-cell of the following type:

$$\mathsf{SY}[\![\alpha, \beta]\!] : \mathsf{EH}_4[\![\alpha, \beta]\!] \to \mathsf{EH}_4^{-1}[\![\beta, \alpha]\!]$$

The syllepsis 5-cell has not been constructed in Catt. However, we have successfully formalized it in $\mathsf{Catt}_{\mathsf{su}}$ due to the reduction in proof complexity in that theory. The formalization is linked above.

We believe this to be the first formalization of the syllepsis as a pure path-theoretic object. Following dissemination of our results, others have now also formalized the syllepsis in other theories [18].

5 Disc trivialization

In the following section, we prove the following structure theorem: in a disc context D^n, up to definitional equality, every term is either a variable, or an iterated identity on a variable. So if we restrict to terms that use *all* variables of D^n (that is, terms which do not factor through a smaller context), then there is exactly *one* definitional equivalence class of term in each dimension $k \geq n$, the normal form of which is the locally-maximal variable d_n, or an iterated identity on this. In this sense, $\mathsf{Catt_{su}}$ trivializes disc contexts.

This provides an interesting point of comparison with work of Batanin, Cisinski and Weber [5], discussed further in Section 1.1, where strictly unital ∞-categories are defined as algebras over a certain operad, defined to be the universal one which trivializes the operations over discs, and which satisfies a unit compatibility property. By showing our theory also trivializes operations over discs, we claim that in this respect, our theory is at least as strict.

Structure Theorem Our structure theorem is stated as follows:

Theorem 45 (Disc trivialization)**.** *Suppose t is valid in D^n. Then t is definitionally equal to a variable, or to the iterated canonical identity on a variable.*

To prove this, we need to establish some technical results about pasting contexts. We say that a substitution $\Delta \vdash \sigma : \Gamma$ is a *variable-to-variable substitution* if for all $x \in \mathsf{var}(\Gamma)$, we have that $x[\![\sigma]\!]$ is again a variable.

Lemma 46. *Let $\mathrm{D}^n \vdash \sigma : \Gamma$ be a valid substitution which is in normal form, and which sends locally-maximal variables of Γ to variables of D^n. Then σ is a variable-to-variable substitution.*

Proof. Let v be a variable of Γ. Then there is some locally-maximal variable w of Γ such that $v = \mathsf{src}^k(w)$ for some $k \in \mathbb{N}$. It follows from the formation rules for substitution that $\mathsf{src}^k(w[\![\sigma]\!]) = \mathsf{src}^k(w)[\![\sigma]\!] \equiv v[\![\sigma]\!]$. Since we are given that $w[\![\sigma]\!]$ is a variable, it follows that $v[\![\sigma]\!]$ is a variable up to definitional equivalence. But since σ is normalized, $v[\![\sigma]\!]$ must be precisely a variable. $\square$

For $k < n$, there are two variable-to-variable substitutions $\mathrm{D}^n \vdash \partial^{\pm}_{k,n} : \mathrm{D}^k$, which map the k-disc context into the appropriate source or target context of D^n. We also have $\mathrm{D}^n \vdash \mathrm{id}_{\mathrm{D}^n} : \mathrm{D}^n$, the identity substitution. We call these *subdisc inclusions.* We now show that every valid variable-to-variable substitution $\mathrm{D}^n \vdash \sigma : \Gamma$ is of this form.

Lemma 47. *Let $\mathrm{D}^n \vdash \sigma : \Gamma$ be a valid variable-to-variable substitution. Then we have $\Gamma \equiv \mathrm{D}^k$ for some $k \leq n$, and σ is a subdisc inclusion.*

Proof. The variables of a pasting context Γ form a globular set $\mathsf{g}(\Gamma)$ in an obvious way, and the substitution well-typedness condition means that a variable-to-variable substitution $\mathrm{D}^n \vdash \sigma : \Gamma$ induces a function of globular sets $\sigma : \mathsf{g}(\Gamma) \to \mathsf{g}(\mathrm{D}^n)$. Suppose for a contradiction Γ is *not* a disc: then it must contain some sub–Dyck word $(\Uparrow (\Downarrow (\Uparrow (\cdots)\, y\, f) \cdots) z\, g \cdots)$, and we have $\mathsf{tgt}(f) = \mathsf{src}(g)$. Then also $\mathsf{tgt}(\sigma(f)) = \mathsf{src}(\sigma(g))$; but now we have a contradiction, since the globular set of a disc does not have any pair of elements related in this way.

We conclude that $\Gamma \equiv \mathrm{D}^k$ for some $k \leq n$. It remains to show that σ is a subdisc inclusion. For this, suppose $k = n$. Then since σ preserves variable dimension, we must have $d_n[\![\sigma]\!] = d_n$, and this extends uniquely to the other variables, since σ is a function of globular sets, and we conclude $\sigma \equiv \mathrm{id}_{\mathrm{D}^n}$. Otherwise, suppose $k < n$. Then we can choose $d_k[\![\sigma]\!] = d_k$ or $d_k[\![\sigma]\!] = d'_k$, and once again, both extend uniquely, yielding $\sigma \equiv \partial^-_{k,n}$ and $\sigma \equiv \partial^+_{k,n}$ respectively. $\square$

Given a valid term t in some context Γ, its *canonical identity* is $\mathbb{1}(t) := \mathbb{1}_{\mathsf{dim}\,\mathsf{ty}(t)}[\![\{\mathsf{ty}(t), t\}]\!]$. Canonical identities can be distinguished from ordinary identities $\mathbb{1}_n[\![\sigma]\!]$ because we do not need to give the dimension subscript, as it can be inferred from the term and the supplied context; because we use round brackets; and because we supply a term as an argument, rather than a substitution. A term is an *iterated canonical identity* if it is of the form $\mathbb{1}^k(t)$, by applying this construction k times for $k > 0$. We now show that if a term is definitionally equal to an ordinary identity $\mathbb{1}_n[\![\sigma]\!]$, it is definitionally equal to a canonical identity.

Lemma 48. *If t is a valid term of Γ with $t = \mathbb{1}_n[\![\ldots, p]\!]$, then $t = \mathbb{1}(p)$.*

Proof. We define $\sigma := [\ldots, p_2, p_2', p_1, p_1', p]$. Because $\Gamma \vdash \sigma : \mathrm{D}^n$ is a valid substitution, it must satisfy the substitution typing conditions up to definitional equality, so we conclude for each $0 < k \leq n$ the following:

$$\begin{aligned} p_k &\equiv d_{n-k}[\![\sigma]\!] \equiv \mathsf{src}^k(d_n)[\![\sigma]\!] = \mathsf{src}^k(d_n[\![\sigma]\!]) \equiv \mathsf{src}^k(p) \\ p_k' &\equiv d_{n-k}'[\![\sigma]\!] \equiv \mathsf{tgt}^k(d_n)[\![\sigma]\!] = \mathsf{tgt}^k(d_n[\![\sigma]\!]) \equiv \mathsf{tgt}^k(p) \end{aligned}$$

We now reason as follows:

$$\begin{aligned} t &\equiv \mathbb{1}_n[\![\ldots, p_2, p_2', p_1, p_1', p]\!] \\ &= \mathbb{1}_n[\![\ldots, \mathsf{src}^2(p), \mathsf{tgt}^2(p), \mathsf{src}(p), \mathsf{tgt}(p), p]\!] \\ &\equiv \mathbb{1}(p) \end{aligned}$$

This completes the proof. □

We are now able to prove Theorem 45.

Proof of Theorem 45. If t is a variable, we are done. Otherwise, t is a coherence term, and we have that $t \equiv \mathsf{coh}\,(\Gamma : U)[\sigma]$. By Corollary 40, we may assume without loss of generality that t is in normal form.

If t is an identity, then by Lemma 48 we know $t = \mathbb{1}(u)$. By induction on dimension, u is therefore either a variable or a iterated identity on a variable, and we are done.

It remains to consider the case that $t \equiv \mathsf{coh}\,(\Gamma : U)[\sigma]$ is not an identity. We will see that this leads to a contradiction. Since t is in normal form, we know that t is not an A-, B-, C-, D- or E-redex, and we use these facts freely below.

First, note that $\mathrm{D}^n \vdash \sigma : \Gamma$ maps locally maximal variables of Γ to non-identity terms of D^n (or else t would be a B-redex), and these terms are in normal form (or else t would be an A-redex). Hence, by induction on subterms, we may assume that σ maps locally-maximal variables to variables. By Lemma 46, it follows that σ is a variable-to-variable substitution, and then from Lemma 47 we conclude that Γ is a disc context D^k with $k \leq n$, and σ is a subdisc inclusion. We therefore conclude that $t \equiv \mathsf{coh}\,(\mathrm{D}^k : u \to_T v)[\sigma]$.

Suppose $\mathsf{dim}\, t = k$. Then u, v must each use all the variables of the respective boundary context, so by induction on subterms, the only possibility is $u = d_{k-1}$ and $v = d_{k-1}'$. Since t is not a C-redex, we conclude that u is in normal form (hence $u \equiv d_{k-1}$), v is in normal form (hence $v \equiv d_{k-1}'$), and T is in normal form (hence $T \equiv S^{k-2}$), and so $t \equiv \mathsf{coh}\,(\mathrm{D}^k : d_{k-1} \to_{S^{k-2}} d_{k-1}')[\sigma]$. But then t would be a D-redex, which is a contradiction.

So we must have $\mathsf{dim}\, t > k$. Then u, v must each use all the variables of D^k, so by induction on subterms, the only possibility is $u = v = \mathbb{1}^{n-k}(d_k)$. But this would mean that t is an E-redex, again giving a contradiction. □

6 Rehydration

In this section, we show that for every $\mathsf{Catt}^{\mathsf{pd}}_{\mathsf{su}}$ term t, we can produce a $\mathsf{Catt}^{\mathsf{pd}}$ term $R(N(t))$, its *rehydrated normal form*, with the property that $t = R(N(t))$. To see why this is useful, consider that while $N(t)$ is in normal form, its sources and targets will not necessarily be. The rehydrated normal form "fixes up" the boundaries recursively, putting them all into rehydrated normal form. This means that if u, v are composable in $\mathsf{Catt}^{\mathsf{pd}}_{\mathsf{su}}$, the terms $R(N(u)), R(N(v))$ are again composable on-the-nose in $\mathsf{Catt}^{\mathsf{pd}}$, and indeed are themselves valid in $\mathsf{Catt}^{\mathsf{pd}}$.

Our strategy is to introduce the following operations on any valid term t, simultaneously by mutual recursion:

- the *rehydration* $R(t)$, which rehydrates all subterms, and then pads the resulting term;
- the *padding* $P(t)$, which composes a term at its boundaries, ensuring all of its sources and targets are in rehydrated normal form;
- the *normalizer* $\phi(t)$, a coherence term which provides an explicit equivalence between t and its rehydrated normal form $R(N(t))$.

We begin with the definition of rehydration.

Definition 49. For a valid $\mathsf{Catt}^{\mathsf{pd}}_{\mathsf{su}}$ term $\Gamma \vdash t : A$, its *rehydration* $R(t)$ is defined as follows:

- $R(x) := x$
- $R\big(\mathsf{coh}\,(\Theta : U)[\sigma]\big) := P\big(\mathsf{coh}\,(\Theta : R(U))[R(\sigma)]\big)$

On valid types and substitution, rehydration is defined by applying term rehydration to all subterms and subtypes.

We next give the definition of padding, which composes a term with normalizers to put all its boundaries into rehydrated normal form. Since these boundaries have strictly smaller dimension than original term, the mutual recursion is well-founded. We use the notation $\mathsf{comp}_{\mathsf{n},\mathsf{k}}$ for the coherence introduced on page 8, and for simplicity list only the locally-maximal arguments of the substitution.

Definition 50. For a valid $\mathsf{Catt}^{\mathsf{pd}}_{\mathsf{su}}$ term $\Gamma \vdash t : A$, its *padding* $P(t)$ is defined by $P(t) := P_{\mathsf{dim}\,A+1}(t)$, and then:

- $P_0(t) := t$
- $$\begin{aligned} P_{k+1}(t) := \mathsf{comp}_{\mathsf{dim}\,A+1,k}\big(&\phi(\mathsf{src}_k(P_k(t))),\\ &P_k(t),\\ &\phi^{-1}(\mathsf{tgt}_k(P_k(t)))\big) \end{aligned}$$

The constructors P_k each "fix up" the corresponding source and targets of their arguments, so $P_k(t)$ is guaranteed to have its j-boundary in rehydrated normal form for all $j < k$.

Finally we give the definition of the normalizer data.

Definition 51. For a valid $\mathsf{Catt}^{\mathsf{pd}}_{\mathsf{su}}$ term $\Gamma \vdash t : A$, its *normalizer* $\phi(t)$ and *inverse normalizer* $\phi^{-1}(t)$ are defined as:

$$\phi(t) := \mathsf{coh}\,\big(\Gamma : R(N(t)) \to_{\mathsf{ty}(t)} t\big)[\mathrm{id}_\Gamma]$$
$$\phi^{-1}(t) := \mathsf{coh}\,\big(\Gamma : t \to_{\mathsf{ty}(t)} R(N(t))\big)[\mathrm{id}_\Gamma]$$

We now arrive at the main result of this section.

Theorem 52. *Let Γ be a pasting context and suppose we have $\Gamma \vdash t : A$ for some* $\mathsf{Catt}^{\mathsf{pd}}_{\mathsf{su}}$ *term t such that* $\mathsf{supp}(t) = \mathsf{FV}(\Gamma)$*. Then the following properties hold:*

1. *$R(N(t))$ is a valid term in* Catt*;*
2. *$R(N(t)) = t$ in* $\mathsf{Catt}_{\mathsf{su}}$*.*

We will prove (1) and (2) by simultaneous induction on the structure of terms as well as their dimension.

Proof of Property 1. It will be sufficient to prove that $R(t)$ is valid in Catt for any t which is in normal form, since the statement clearly follows from this assertion. We therfore proceed by induction on the structure of t assuming that it (and therefore all of its subterms) are in normal form.

If t is a variable then $R(t) = t$ for which the statement is clearly true. Otherwise $t = \mathsf{coh}\,(\Theta : U)[\sigma]$ and we have by definition

$$R(t) = P(\mathsf{coh}\,(\Theta : R(U))[R(\sigma)])$$

We will first show that the subterm

$$t' := \mathsf{coh}\,(\Theta : R(U))[R(\sigma)]$$

is itself valid in Catt and then argue that applying the padding operation P preserves this validity. We first note that since a straightforward induction on the definition gives that $\mathsf{supp}(R(t)) = \mathsf{supp}(t)$, it follows that $R(U)$ will itself satisfy the free variable condition for coh term formation and that, by the induction hypothesis, $R(U)$ is a valid Catt type.

Now, the terms comprising the substituion $R(\sigma)$ are themselves also valid by induction. This is not, in itself, enough to ensure that $R(\sigma)$ is a valid Catt substitution. The additional conditions which must be verified are all of the form

$$\mathsf{src}_k(R(u)) \equiv \mathsf{tgt}_l(R(v))$$

for inferred sources and targets of some terms $u, v \in \sigma$. But in any such case, we must have $\mathsf{src}_k(u) = \mathsf{tgt}_l(v)$ in $\mathsf{Catt}_{\mathsf{su}}$ by the assumption that the original term t was valid. It follows that

$$R(N(\mathsf{src}_k(u))) \equiv R(N(\mathsf{tgt}_l(v)))$$

using that u and v are in normal form.

We now claim that we have the equations

$$\begin{aligned} \mathsf{src}_k(R(u)) &\equiv R(N(\mathsf{src}_k(u))) \\ \mathsf{tgt}_l(R(u)) &\equiv R(N(\mathsf{tgt}_k(v))) \end{aligned}$$

which, combined with the observation above will complete the claim that $R(\sigma)$ is a valid Catt substitution.

To see this, we calculate:

$$\begin{aligned} \mathsf{src}(R(u)) \equiv \mathsf{src}\Big(\mathsf{comp}_{\mathsf{dim}\,A+1,\mathsf{dim}A}\big(&\phi(\mathsf{src}(P_{\mathsf{dim}A}(u))), \\ &P_{\mathsf{dim}A}(N(t)), \\ &\phi^{-1}(\mathsf{tgt}(P_{\mathsf{dim}A}(u)))\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&\equiv \mathsf{src}(\phi(\mathsf{src}(P_{\mathsf{dim}A}(u)))) \\
&\equiv \mathsf{src}(\mathsf{coh}\,(\partial^-(\Gamma) : R(N(\mathsf{src}(P_{\mathsf{dim}A}(u)))) \\
&\qquad\qquad\qquad \to \mathsf{src}(P_{\mathsf{dim}A}(u)))[\mathrm{id}_\Gamma]) \\
&\equiv R(N(\mathsf{src}(P_{\mathsf{dim}A}(u)))) \\
&\equiv R(N(\mathsf{src}(N(P_{\mathsf{dim}A}(u))))) \\
&\equiv R(N(\mathsf{src}(u)))
\end{aligned}
$$

This second to last step follows since $N(\mathsf{src}(N(t))) \equiv N(\mathsf{src}(t))$ as a consequence of definitional equality preserving typing. The last step follows by an application of property (2) of the theorem, which is valid in this case because $\mathsf{src}(u)$ must be of strictly smaller dimension. The result for src_k follows by a simple induction on k. The case for tgt_k and v is similar.

We have now shown that t' is indeed a valid Catt term and we note for below that it also follows that all of its inferred sources and targets are valid Catt terms as well.

We now argue that $P(t')$ is valid in Catt. We prove this by induction on the parameter k of the padding construction:

$$
\begin{aligned}
P_{k+1}(t') \equiv \mathsf{comp}_{\mathsf{dim\,A+1,k}}\big(&\phi(\mathsf{src}_k(P_k(t'))), \\
&P_k(t'), \\
&\phi^{-1}(\mathsf{tgt}_k(P_k(t')))\big)
\end{aligned}
$$

The composition operations $\mathsf{comp}_{\mathsf{d,k}}$ are certainly valid Catt terms, as elementary syntactic constructions. By induction, the subterm $P_k(t')$ is valid in Catt. The normalizers in this expression are being computed for terms $\mathsf{src}_k(P_k(t'))$ and $\mathsf{tgt}_k(P_k(t'))$. While these terms are not themselves in rehydrated normal form, their inferred types $\mathsf{ty}(\mathsf{src}_k(P_k(t')))$, $\mathsf{ty}(\mathsf{tgt}_k(P_k(t')))$ *will* be in rehydrated normal form, thanks to the remark following Definition 50.

To complete the proof, we must therefore show that if u is some valid term with $\mathsf{ty}(u)$ in rehydrated normal form, then $\phi(u)$ and $\phi^{-1}(u)$ are valid in Catt. We consider the definition of $\phi(u)$:

$$
\phi(u) := \mathsf{coh}\left(\Gamma : R(N(u)) \to_{\mathsf{ty}(u)} u\right)[\mathrm{id}_\Gamma]
$$

Since in all cases of interest $\mathsf{dim}\, u < \mathsf{dim}\, t$, we know by induction on dimension that $R(N(u))$ is valid in Catt. We know u arises as an inferred source or target of t', and hence is valid in Catt. We also know $\mathsf{ty}(u)$ is already in rehydrated normal form, and hence $\mathsf{ty}(u) \equiv \mathsf{ty}(R(N(u)))$. So $\phi(u)$ is valid in Catt, as is $\phi^{-1}(u)$. ☐

Proof of Property 2. For a variable, this is immediate, and we therefore focus on coherence terms. We may assume that the result holds on all terms u of strictly smaller dimension than t. This allows us to show that such terms have normalizers which themselves normalize to identities:

$$
\begin{aligned}
\phi(u) &\equiv \mathsf{coh}\,(\Gamma : u \to_{\mathsf{ty}(u)} R(N(u)))[\mathrm{id}_\Gamma] \\
&\overset{\mathrm{C}}{\rightsquigarrow}_{\mathsf{rt}} \mathsf{coh}\,(\Gamma : N(u) \to_{N(\mathsf{ty}(u))} N(u))[\mathrm{id}_\Gamma] \\
&\overset{\mathrm{E}}{\rightsquigarrow} \mathbb{1}_{\mathsf{dim}\,t}[\![\{N(\mathsf{ty}(u)), N(u)\}]\!]
\end{aligned}
$$

Note that we use the equation $N(R(N(u))) \equiv N(u)$ in the first step, this being a consequence of the conclusion of the theorem applied to u, which is of smaller dimension than t.

Now, since paddings are constructed from the normalizers of terms of strictly smaller dimension (namely the sources and targets of t), we may use the previous result to show that paddings normalize to to the term being padded. This we prove by induction on the parameter k in the definition of the padding composite P_k:

$$\begin{aligned} P_{k+1}(t) &\equiv \mathsf{comp_{n,k}}(\phi(\mathsf{src}_k(t)), P_k(t), \phi(\mathsf{tgt}_k(t))) \\ &\overset{\text{A}}{\rightsquigarrow}_{\mathsf{t}} \mathsf{comp_{n,k}}(\mathbb{1}_k[\![\tau]\!], t, \mathbb{1}_k[\![\tau]\!]) \\ &\overset{\text{B}}{\rightsquigarrow}_{\mathsf{t}} \mathsf{coh}\,(D^n : d_n \to_{S^{n-1}} d_n)[t] \\ &\overset{\text{D}}{\rightsquigarrow} t \end{aligned}$$

where $n := \dim t$.

To finish the claim, we argue by induction on the structure of t. As in the proof of (1), we may as well suppose that t is in normal form. Moreover, we may also use the fact that $R(t)$ is valid in Catt and therfore in $\mathsf{Catt_{su}}$ to promote the reductions described above to definitional equalities. Then (ignoring the trivial case of variables) we have:

$$\begin{aligned} R(t) &\equiv R\big(\mathsf{coh}\,(\Theta : U)[\sigma]\big) \\ &\equiv P\big(\mathsf{coh}\,(\Theta : R(U))[R(\sigma)]\big) \\ &= \mathsf{coh}\,(\Theta : R(U))[R(\sigma)] \\ &= \mathsf{coh}\,(\Theta : U)[\sigma] \end{aligned}$$

This completes the proof. □

With Theorem 52 in hand, we are able to conclude with the result promised in Section 2.3.

Corollary 53. *The functor $K^* : \mathcal{C}at_\infty^{\mathrm{su}} \to \mathcal{C}at_\infty$ is fully faithful.*

Proof. It is a well-known elementary observation (cf [3, p.47]) that if a functor $F : \mathsf{A} \to \mathsf{B}$ is essentially surjective on objects and full, then for any category C, the induced precomposition functor $- \circ F : \mathsf{Hom}(\mathsf{B}, \mathsf{C}) \to \mathsf{Hom}(\mathsf{A}, \mathsf{C})$ between functor categories is fully faithful.

Now, we have already seen that the functor K is the identity on objects. Moreover, the argument in the proof of Theorem 52, shows that for any σ which is a valid substitution in $\mathsf{Catt_{su}}$, $R(N(\sigma))$ is a valid substitution in Catt and moreover that $R(N(\sigma)) = \sigma$. This shows that the functor K is full.

As $\mathcal{C}at_\infty^{\mathrm{su}}$ and $\mathcal{C}at_\infty$ are full subcategories of the presheaf categories $\mathsf{Hom}((\mathsf{Catt}_{\mathsf{su}}^{\mathsf{pd}})^{\mathrm{op}}, \mathsf{Set})$ and $\mathsf{Hom}((\mathsf{Catt}^{\mathsf{pd}})^{\mathrm{op}}, \mathsf{Set})$, respectively, and since precomposition with the full functor $K : \mathsf{Catt}^{\mathsf{pd}} \to \mathsf{Catt}_{\mathsf{su}}^{\mathsf{pd}}$ preserves these full subcategories, the result follows. □

In other words, it is a *property* of a given ∞-category to be strictly unital; if there exists a lift of a given ∞-category $\mathsf{C} \in \mathcal{C}at_\infty$ to a strictly unital one $\mathsf{C}' \in \mathcal{C}at_\infty^{\mathrm{su}}$, it is necessarily unique up to isomorphism, and hence a weak ∞-category can admit at most one strictly unital structure.

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A The Eckmann-Hilton term

Here we give the normalized Eckmann-Hilton term in $\mathsf{Catt}_{\mathsf{su}}$, as discussed in Example 43. This is available at the following link:

`http://github.com/ericfinster/catt.io/tree/v0.1`
`examples/example_4_3_cattsu.catt`

For readability, we omit implicit arguments in the following.

$$
\begin{array}{l}
\mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(g:x\to y)(\alpha:f\to g)\\
\qquad\qquad (z:\star)(h:y\to z)(i:y\to z)(\beta:h\to i):\\
\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(g:x\to y)(\alpha:f\to g)\\
\qquad\qquad\qquad (h:x\to y)(\beta:g\to h):f\to h)[\\
\qquad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(g:x\to y)(\alpha:f\to g)(z:\star)(h:y\to z):\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[f,h]\\
\qquad\quad \to\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[g,h])[\alpha,h],\\
\qquad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z)(h:y\to z)(\alpha:g\to h):\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[f,g]\\
\qquad\quad \to\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[f,h])[g,\beta]\\
\quad]\\
\quad \to\\
\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(g:x\to y)(\alpha:f\to g)\\
\qquad\qquad\qquad (h:x\to y)(\beta:g\to h):f\to h)[\\
\qquad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z)(h:y\to z)(\alpha:g\to h):\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[f,g]\\
\qquad\quad \to\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[f,h])[f,\beta],\\
\qquad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(g:x\to y)(\alpha:f\to g)(z:\star)(h:y\to z):\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[f,h]\\
\qquad\quad \to\\
\qquad\quad \mathsf{coh}\,((x:\star)(y:\star)(f:x\to y)(z:\star)(g:y\to z):x\to z)[g,h])[\alpha,i]\\
\quad])[a,b]
\end{array}
$$