A continuation monad for quantum effects in recursive programs

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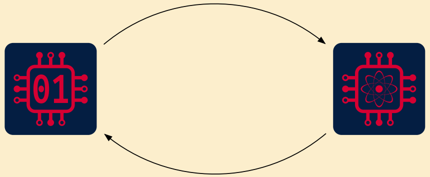
Birmingham Theory Seminar

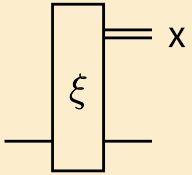
Quantum Software Lab

Overview

- A model of hybrid quantum-classical computing



- Quantum instruments as a computational effect (monad structure).



- Extending this monad to complete partial orders to model recursion.

$$\mathrm{fix}(f) = \sup(\perp \leq f(\perp) \leq f^2(\perp) \leq \ldots)$$

What is hybrid quantum-classical computing?

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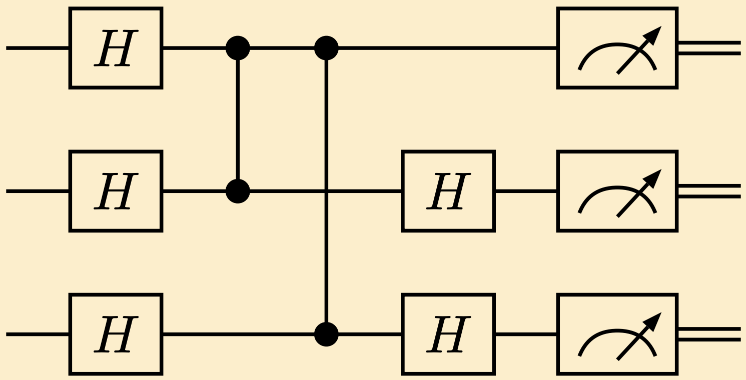
Quantum computing is typically described using *circuit diagrams*.

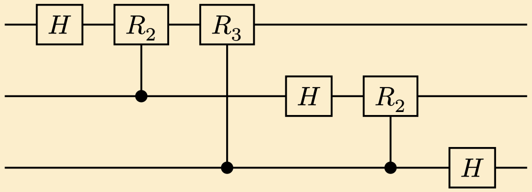


Adding classical control

Circuit generation

```
def qft(qs):
    if not qs:
        return
    H(qs[0])
    for q in qs[1:]:
        ...
    qft(qs[1:])
```

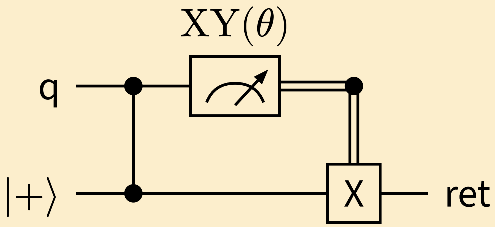
↓



Feedforward

```
def mbqc(theta, q):
    ret = ket('+')
    CZ(q, ret)
    if xy_measure(theta, q):
        X(ret)
    return ret
```

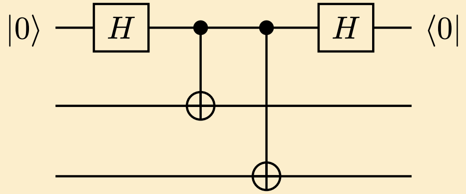
↓



Postselection

```
def post_sel(q1, q2):
    a = H(ket(0))
    CX(a, q1)
    CX(a, q2)
    H(a)
    if measure(a):
        fail()
```

↓



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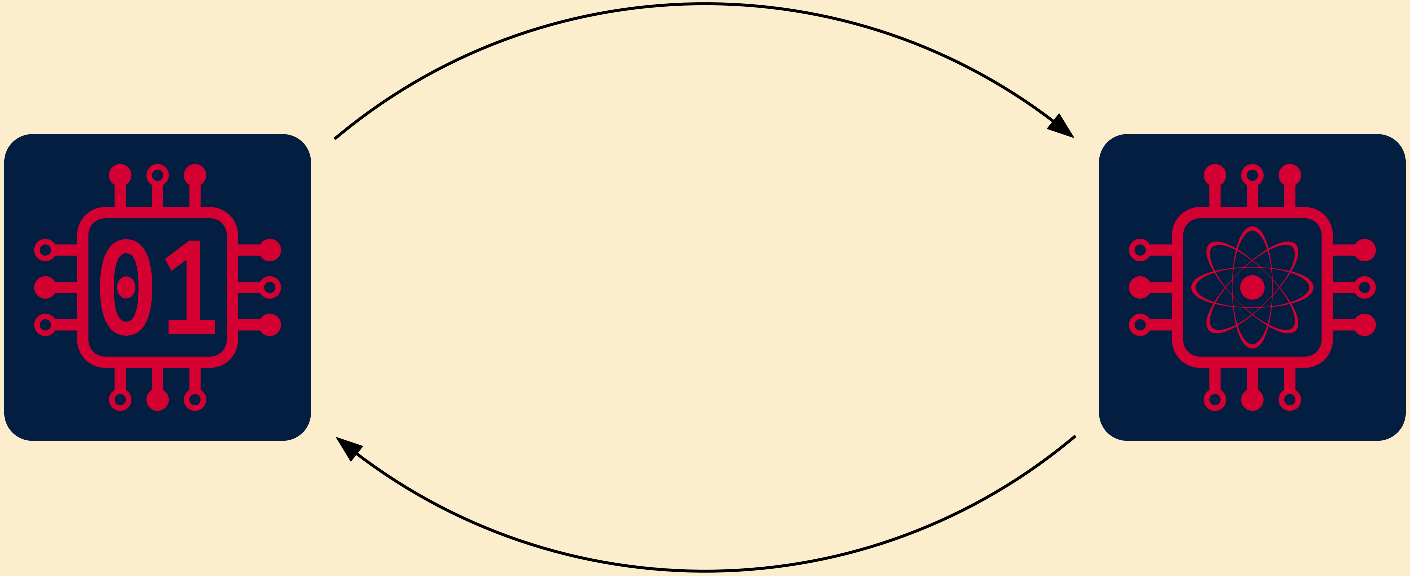

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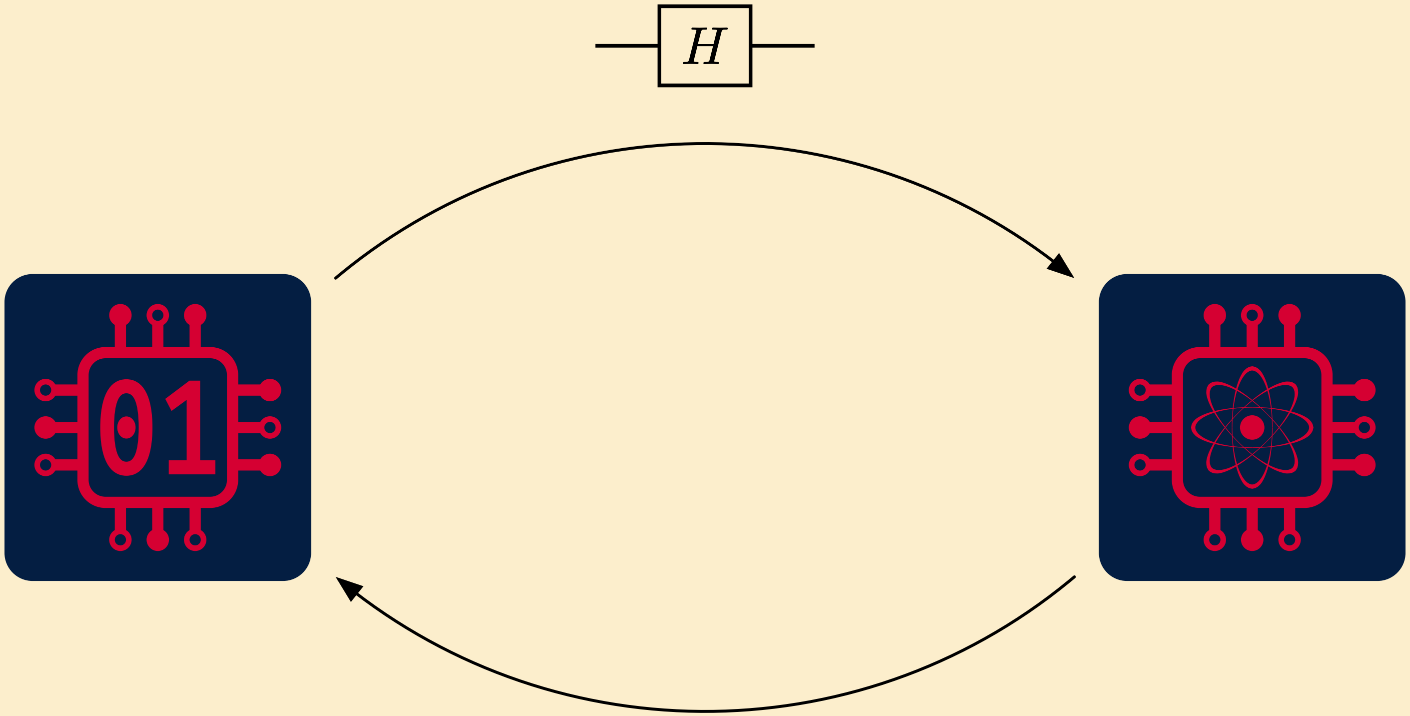

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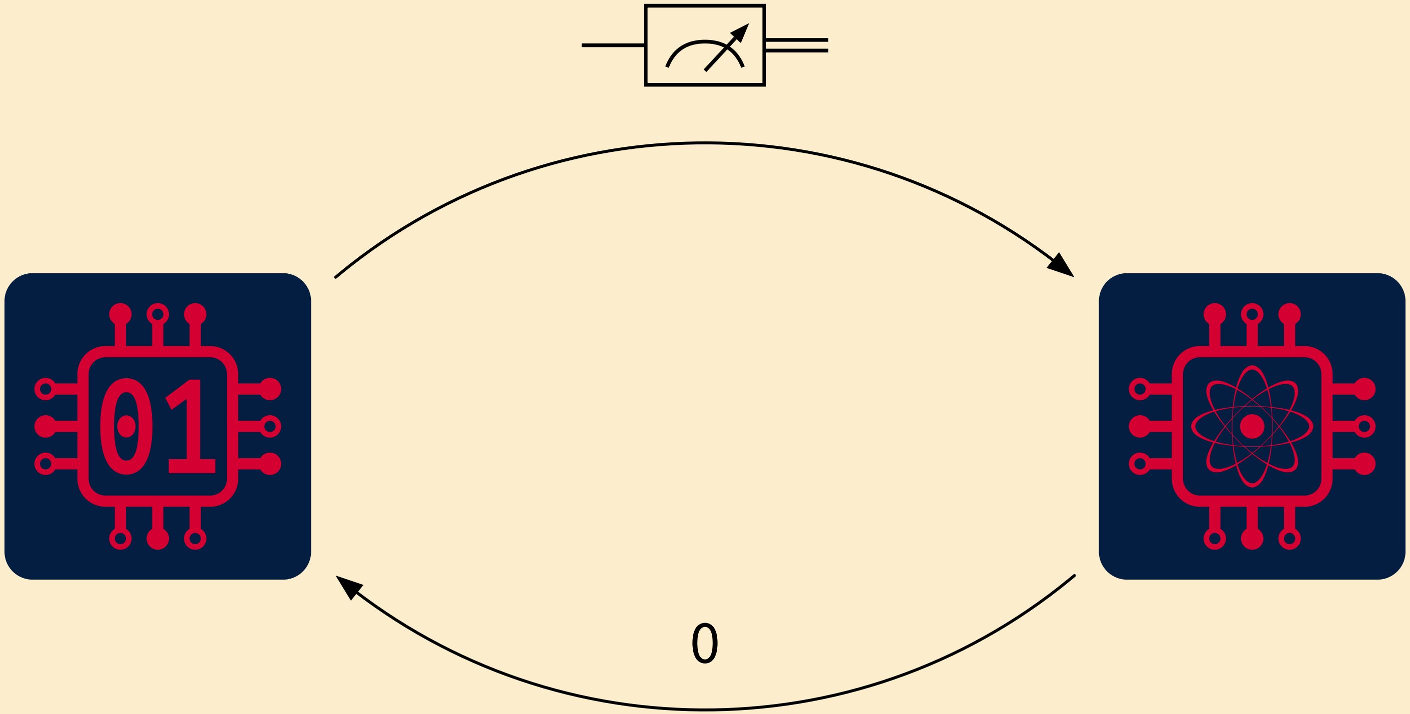

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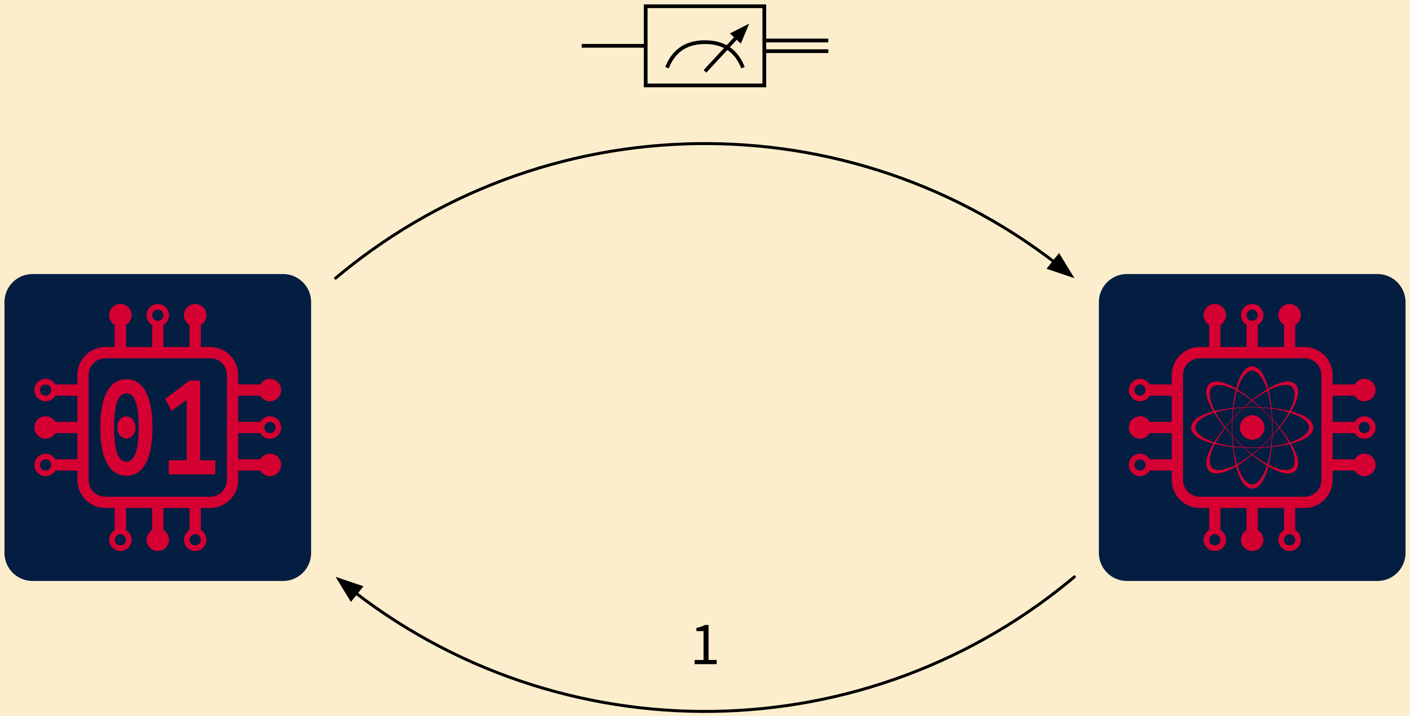

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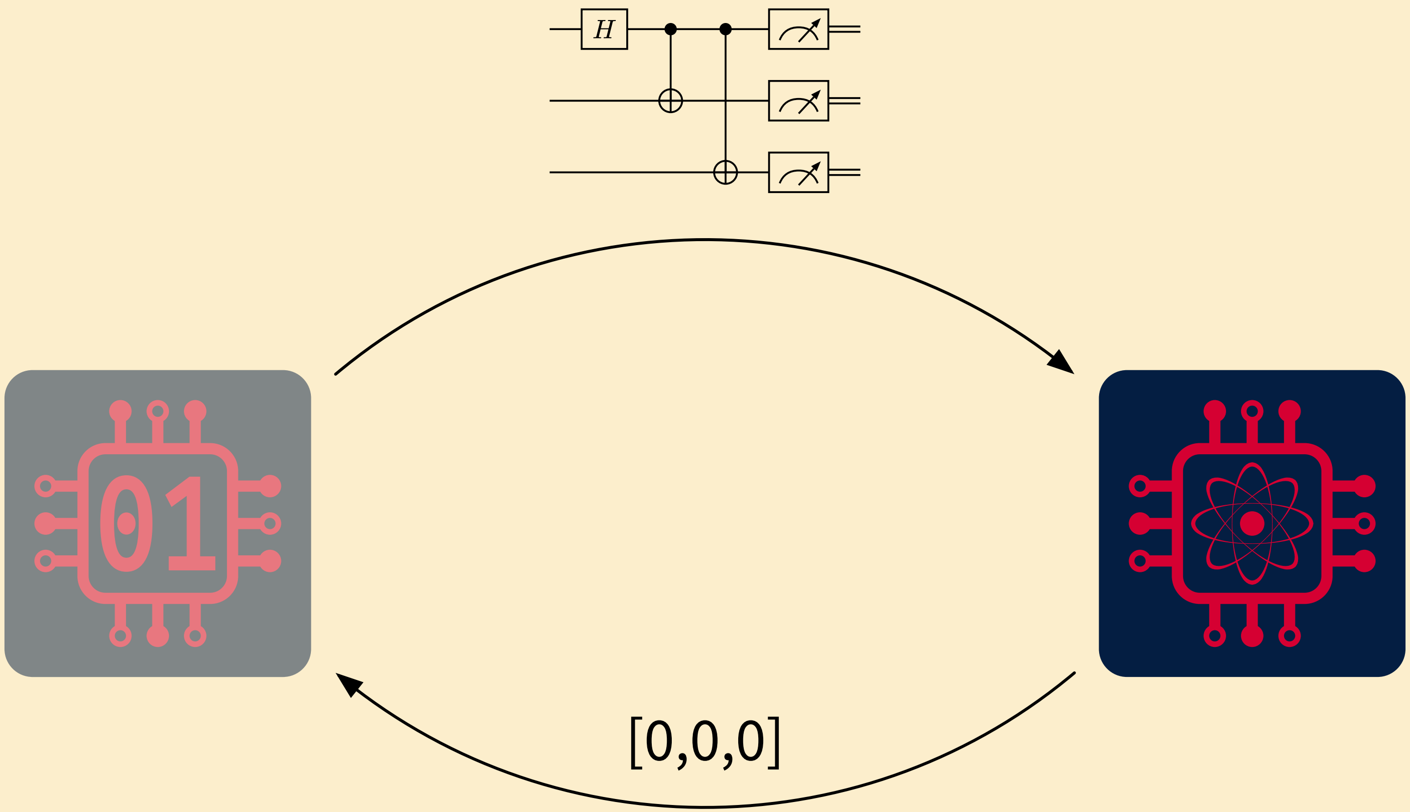

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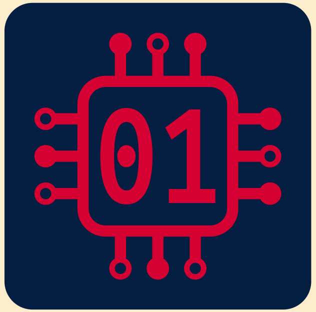

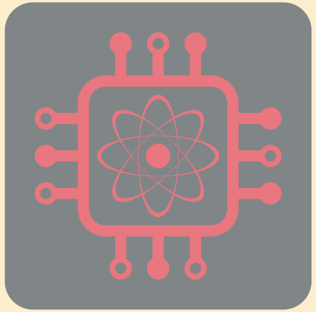

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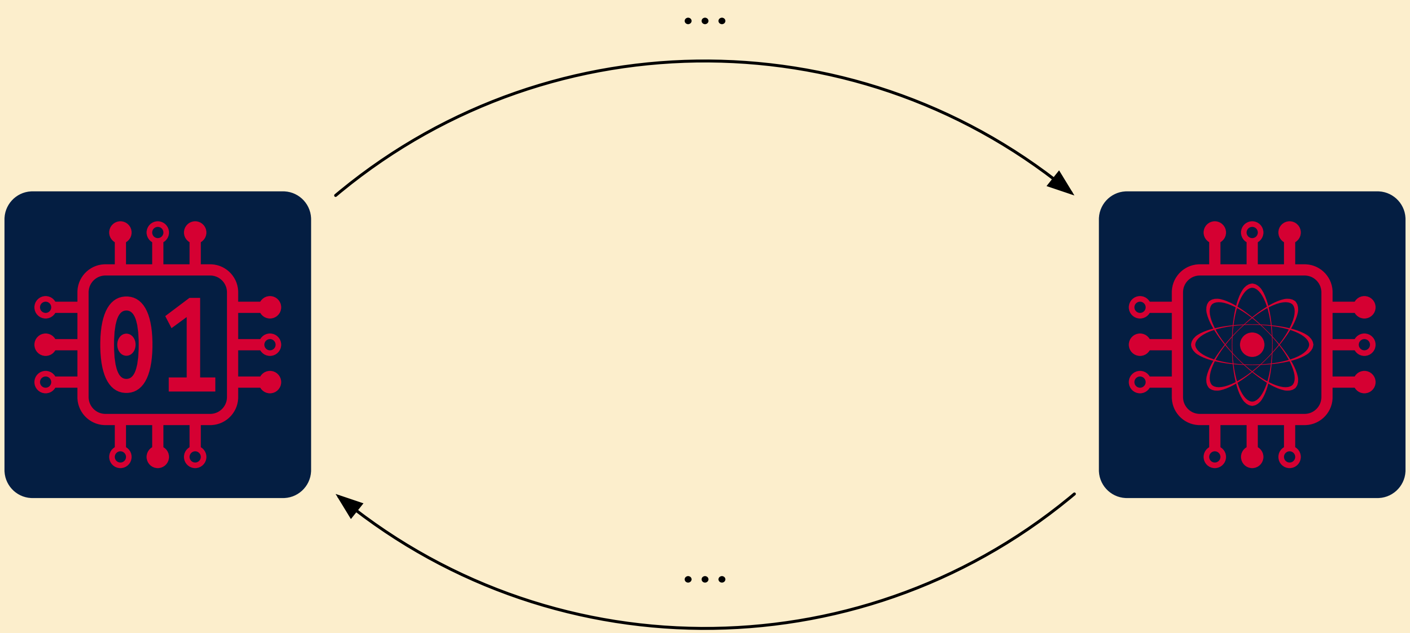


What does such a system do?

How can we model this system?

Computational effects

"An effect is anything a function does beyond mapping inputs to outputs"

Common effects include:

- IO
- Failure
- Non-determinism
- Probabilistic choice

```
def effectful_function(i: int):
    if random():
        return i
    else:
        return -i
```

Probabilistic computation

Computations with randomness can be represented with *distributions*.

```
def effectful_function(i: int):
    if random():
        return i
    else:
        return -i
```

This takes integer i to $\begin{cases} i \text{ with probability } 0.5 \\ -i \text{ with probability } 0.5 \end{cases}$

A distribution on X can be represented as a function $X \to [0, 1]$.

Quantum states and channels

Quantum computation can also modify the quantum data on the computer.

Concept	Mathematical model
Quantum state	Operator on a Hilbert space which is: • Positive semi-definite • Trace 1
Quantum channel (QChannel)	Map between operator spaces which is: • Completely positive • Trace non-increasing
Normalised Quantum channel	Trace-preserving quantum channel

Quantum instruments

The classical output of measurement is not captured by channels.

Quantum instruments extend channels with classical output.

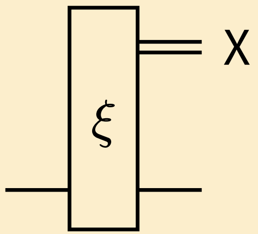
We split the channel over classical outputs $x \in X$.

Definition: (Finite) quantum instrument

A *quantum instrument* on a set X is a map:

$$\xi : X \to \text{QChannel}$$

s.t. $\xi(x) \neq 0$ for finitely many x and $\sum_x \xi(x)$ is normalised.



Example: Gate as an instrument

A quantum gate has no classical output.

We represent this as the classical outcome set $X = \{*\}$.

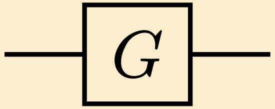
Given a gate G, we get instrument:

$$\overline{G} : \{*\} \to \mathsf{QChannel}$$

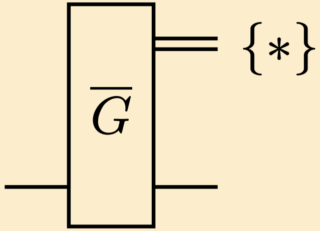
$$\overline{G}(*) = [\![G]\!]$$

Where $[\![G]\!]$ is the channel describing the application of G.

Gate:



Instrument:



Example: Measurement as an instrument

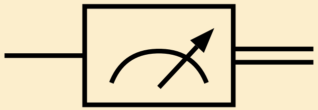
Measurement has 2 classical outcomes,
so let $X = \text{bool} = \{\text{False}, \text{True}\}$.

$$M : \text{bool} \to \text{QChannel}$$
$$M(\text{False}) = P_0$$
$$M(\text{True}) = P_1$$

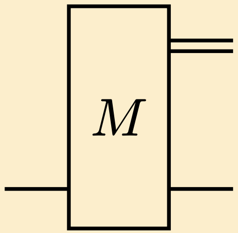
Where P_0 projects to the 0 state and P_1 projects to the 1 state.

$$P_0 + P_1 \text{ is normalised}$$

Measurement:



Instrument:



Computational effects via monads

Computational effects are commonly modelled with monads.

Monad M takes an X to $M(X)$ of computations returning X.

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Unit

$\eta_X : X \to M(X)$

Includes non-effectful computation

```
def no_effect():
    return 1
```

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```
def no_effect():
    return 1
```

Extension

$f : X \to M(Y) \Rightarrow f^* : M(X) \to M(Y)$

Composition of effectful functions.

```
def effectful(x):
    return x + random()
a = effectful(1)
b = effectful(a)
```

Example: distribution monad

$$P(X) = \left\{ p : X \to [0,1] \mid p(x) \neq 0 \text{ for finitely many } x \text{ and} \sum_x p(x) = 1 \right\}$$

Unit

$$\eta_X : X \to P(X)$$

$$\eta_X(x) = y \mapsto \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$$

Extension

Given $f : X \to P(Y)$ obtain:

$$f^* : P(X) \to P(Y)$$

$$f^*(p) = y \mapsto \sum_x p(x) \cdot f(x)(y)$$

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Primitives

$\mathsf{random} : P(\mathsf{bool})$ $\qquad$ $\mathsf{random} = b \mapsto 0.5$

The quantum instrument monad

$$Q(X) = \left\{ \xi : X \to \mathsf{QChannel} \mid \xi(x) \neq 0 \text{ for finitely many } x \right.$$
$$\left. \text{and } \sum_x \xi(x) \text{ is Normalised} \right\}$$

Unit

$$\eta_X : X \to Q(X)$$

$$\eta_X(x) = y \mapsto \begin{cases} \mathbb{1} \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$$

Extension

Given $f : X \to Q(Y)$ obtain:

$$f^* : Q(X) \to Q(Y)$$

$$f^*(\xi) = y \mapsto \sum_x \xi(x) \circ f(x)(y)$$

Primitives $\overline{U} : Q(\{*\})$ $M : Q(\mathsf{bool})$

Theorem

Q is a monad on $\mathbf{Set}$.
Further, P is a submonad of Q.

Given a distribution $p : X \to [0,1]$, we get a quantum instrument:

$$q : X \to \mathsf{QChannel}, \quad q(x) = p(x) \cdot \mathbb{1}$$

Example: Reset

One way to reset a qubit to $|0\rangle$ is the following program:

```
def reset(q):
    b = measure(q)
    if b:
        X(q)
```

Lines 3-4 are given by:

$$f : \mathsf{bool} \to Q(\{*\}) \quad f(\mathsf{False}) = \eta(*) \quad f(\mathsf{True}) = \overline{X}$$

with the whole program being $f^*(M)$.

$$\begin{aligned} f^*(M)(*) &= P_0 \circ \mathbb{1} + P_1 \circ [\![X]\!] \\ &= \rho \mapsto \mathrm{tr}(\rho) \cdot \text{0-state} \\ &= [\![\text{Reset}]\!] \end{aligned}$$

Modelling recursion

Many quantum programs contain unbounded recursion/loops.

Recursive definitions are transformed to fixpoints.

DCPOs (Directed complete partial orders), always have least fixpoints.

$$\mathrm{fix}(f) = \sup \begin{pmatrix} \vdots \\ \rotatebox{90}{$\geq$} \\ f^n(\bot) \\ \rotatebox{90}{$\geq$} \\ \vdots \\ \rotatebox{90}{$\geq$} \\ f^3(\bot) \\ \rotatebox{90}{$\geq$} \\ f^2(\bot) \\ \rotatebox{90}{$\geq$} \\ f^1(\bot) \\ \rotatebox{90}{$\geq$} \\ \bot \end{pmatrix}$$

Is there a $\mathbf{DCPO}$-monad $\mathcal{Q}$ of quantum instruments?

Theorem (Cho 2014): QChannel forms a DCPO

Consequence 1: normalisation is too strong

For each DCPO X, $\mathcal{Q}(X)$ should have a bottom element $\perp$.

If X has a bottom element $\perp_X$, then we could let:

$$\perp = \eta(\perp_X)$$

We don't care about the result quantum state from divergence. Instead, we want:

$$\begin{aligned} &\perp : X \to \text{QChannel} \\ &\perp(x) = 0 \end{aligned}$$

Forcing the normalisation condition to be removed.

Consequence 2: finite does not suffice

Consider the following program:

```
def looping(q):
    n = 0
    while measure(q):
        H(q)
        n += 1
    return n
```

This program has infinitely many possible return values.

Therefore finite quantum instruments are not sufficient.

Consequence 3: pointwise does not (seem to) suffice

Let $x_0 \leq x_1 \leq \ldots$ be a chain in X, with $\sup(x_i) = x$.

For the unit of the monad η_X to be continuous, we must have:

$$\eta_X(x_0) \leq \eta_X(x_1) \leq \ldots$$

and

$$\sup(\eta_X(x_i)) = \eta_X(x)$$

Defining an ordering pointwise does not seem to suffice for this to be true.

Comparison with probabilistic power-domain

Quantum can often be seen as an extension of classical probability.

Probablistic

$\mathcal{P}(X) = \{v : \mathcal{O}(X) \to [0,1]\}$

s.t. $\quad v(X) \leq 1$

and

Quantum

$\mathcal{Q}(X) = \{v : \mathcal{O}(X) \to \text{QChannel}\}$

s.t. $\quad v(X)$ is a channel

$$v(U \cup V) + v(U \cap V) = v(U) + v(V)$$

$$v(\emptyset) = 0$$

$$U \subseteq V \Rightarrow v(U) \leq v(V)$$

These rules make v a *valuation*.

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Let

$$v \leq v' := \forall U.\ v(U) \leq v'(U)$$

$\mathcal{P}(X)$ and $\mathcal{Q}(X)$ are DCPOs

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s.t. $\quad v(X) \leq 1$

$\eta : X \Rightarrow \mathcal{P}(X)$

$\eta_X(x)(U) = \begin{cases} 1 \text{ if } x \in U \\ 0 \text{ otherwise} \end{cases}$

Quantum

$\mathcal{Q}(X) = \{v : \mathcal{O}(X) \to \text{QChannel}\}$

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$\eta : X \Rightarrow \mathcal{Q}(X)$

$\eta_X(x)(U) = \begin{cases} \mathbb{1} \text{ if } x \in U \\ 0 \text{ otherwise} \end{cases}$

$$x \leq y \Rightarrow \eta_X(x) \leq \eta_X(y) \quad (\Leftrightarrow \forall U.\ \eta_X(x)(U) \leq \eta_X(y)(U))$$

$$\sup(\eta_X(x_i)) = \eta_X(\sup(x_i))$$

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$$f : X \to \mathcal{P}(Y)$$

$$f^* : \mathcal{P}(X) \to \mathcal{P}(Y)$$

$$f^*(v)(U) = \int_{x \in X} f(x)(U) \ \mathrm{d}v$$

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s.t. $\quad v(X)$ is a channel

$$f : X \to \mathcal{Q}(Y)$$

$$f^* : \mathcal{Q}(X) \to \mathcal{Q}(Y)$$

$$f^*(v)(U) = \quad ?$$

Integration attempt 1: copy classical approach

- The valuation gives definition on indicators:

$$\int \chi(U)\,\mathrm{d}v = v(U)$$

- Extend by linearity to sums of indicators:

$$\int \sum \lambda_i \chi(U_i)\,\mathrm{d}v = \sum \lambda_i v(U_i)$$

- Extend by monotonicity to all functions:

$$\int f\,\mathrm{d}v = \sup\left\{ \int g\,\mathrm{d}v \mid g \leq f, g \text{ simple} \right\}$$

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Channel-valued functions don’t seem to be monotone limits of sums of indicators.

Critically, QChannel is not a lattice.

Integration attempt 2: approximate valuation

Instead of approximating the integrand we could approximate the valuation.

- Define:

$$\int f \, \mathrm{d}\eta(x) = f(x)$$

- Extend by linearity:

$$\int f \, \mathrm{d}\Big(\sum \lambda_i \eta(x_i)\Big) = \sum \lambda_i f(x_i)$$

- Extend by monotonicity.

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- Extend by monotonicity.

The integral on (finite) valuations needs to be continuous to extend by monotonicity.

Proofs of this in the probabilistic case use max-flow min-cut theorems which don't extend from $\mathbb{R}$ to QChannel.

Integration attempt 3: densities

If the valuation v can be written as:

$$v(U) = \int f_v \, \mathrm{d}\Delta_v$$

for probability valuation Δ_v and function f_v, then we can define:

$$\int f \, \mathrm{d}v = \int_x f_v(x) \circ f(x) \, \mathrm{d}\Delta_v$$

This technique seems to only work for measures.
It is unclear how this behaves with respect to the order on valuations.

Axiomatising integration

A valuation v tells us the value of:

$$\int \chi(U)\,\mathrm{d}v$$

for each U, with integration extending this construction to all f.

What if we instead require that v comes pre-equipped with an integral definition:

$$\int f\,\mathrm{d}v$$

Which is convex and preserves monotone limits?

$$\mathcal{Q}(X) = (X \to \mathsf{QChannel}) \to \mathsf{QChannel}$$

$$\mathcal{Q}(X) = (X \to \mathsf{QChannel}) \to \mathsf{QChannel}$$

$$\mathrm{CPS}(X) = (X \to \mathrm{Ans} \qquad) \to \mathrm{Ans}$$

A continuation monad

Quantum orchestra monad

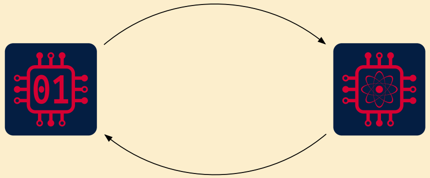
For a DCPO X, let:

$$\mathcal{Q}(X) = \underbrace{\left(\underbrace{X \to \mathsf{QChannel}}_{\text{continuous}}\right) \to \mathsf{QChannel}}_{\text{continuous} + \text{convex}}$$

With unit and extension given identically to the CPS monad.

Summary

- We described a model of hybrid quantum-classical computing through effects.



- We show that quantum instruments form a $\mathbf{Set}$-monad.
- We extend this monad to a $\mathbf{DCPO}$-monad through continuations.

$$\mathcal{Q}(X) = \underbrace{\left(\underbrace{X \to \mathsf{QChannel}}_{\text{continuous}}\right) \to \mathsf{QChannel}}_{\text{continuous + convex}}$$

Pure quantum computation

Measurement free quantum computations are often referred to as pure.

Qubits are represented by the Hilbert space $\mathbb{C}^2$ with basis $\{|0\rangle, |1\rangle\}$.

An n-qubit circuit is represented by a unitary map:

$$U : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$$

Example: Hadamard gate H

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Measurement

Measurement destructively converts quantum states to classical data.

$$M(\alpha|0\rangle + \beta|1\rangle) = \begin{cases} |0\rangle \text{ with probability } |\alpha|^2 \\ |1\rangle \text{ with probability } |\beta|^2 \end{cases}$$

A pure state is converted to a probabilistic ensemble of pure states.

Are pure states the right tool to understand measurement?

No, we instead move to mixed states.

Mixed states

Mixed states are represented through *density matrices*.

A mixed state in H is a trace 1 positive map $\rho : H \to H$

Probabilistic ensembles	Density matrices
Pure states are mixed states	$v \mapsto \|\alpha\rangle\langle\alpha\|v\rangle$ (written $\|\alpha\rangle\langle\alpha\|$)
Probabilities sum to 1	$\mathrm{tr}(\rho) = 1$
Probabilities are positive	ρ is a positive map
Allows convex combinations	$\alpha\rho + \beta\rho'$ is a density matrix

Quantum channels

Quantum computation on mixed states is given by quantum channels.

A *quantum channel* is a completely positive trace-preserving map on $B(H)$.

Unitaries

If $U : H \to H$ is unitary:
Then $\rho \mapsto U\rho U^\dagger$ is a channel.

$$|\alpha\rangle\langle\alpha| \mapsto U|\alpha\rangle\langle\alpha|U^\dagger = |U\alpha\rangle\langle U\alpha|$$

Measurement

What about the classical output?

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Measurement

$$\begin{aligned}\alpha|0\rangle + \beta|1\rangle \mapsto |\alpha|^2\ |0\rangle\langle 0| \\ + |\beta|^2\ |1\rangle\langle 1|\end{aligned}$$

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$$|\alpha\rangle\langle\alpha| \mapsto U|\alpha\rangle\langle\alpha|U^\dagger = |U\alpha\rangle\langle U\alpha|$$

Measurement

$$\begin{aligned}\rho \mapsto\ &\langle 0|\rho|0\rangle|0\rangle\langle 0|\\ &+\langle 1|\rho|1\rangle|1\rangle\langle 1|\end{aligned}$$

What about the classical output?