Biased whisker-based composition in higher categories

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Outline

1. Globular sets
2. Composition in Globular sets
3. Whisker-based composition
A regular (1-)category consists of objects and arrows. In higher category theory we expand this to allow arrows of higher dimensions between lower dimensional arrows. What should these higher dimensional arrows look like?
Globular sets are one natural shape of higher categories:
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**Ordinary 1-categories**

\[
\begin{array}{c}
C_1 \\
\downarrow s \quad \downarrow t \\
C_0
\end{array}
\]
Globular sets are one natural shape of higher categories:

Ordinary 1-categories

\[
\begin{align*}
C_1 & \xrightarrow{s} C_0 \\
| & \quad | \\
\downarrow s & \quad \downarrow t \\
C_0 & \quad C_0
\end{align*}
\]

Globular sets

\[
\begin{align*}
\cdots & \xrightarrow{G_2} G_0 \\
| & \quad | \\
\downarrow s_1 & \quad \downarrow t_1 \\
G_1 & \quad G_1 \\
| & \quad | \\
\downarrow s_0 & \quad \downarrow t_0 \\
G_0 & \quad G_0
\end{align*}
\]
Globular sets can be defined in a few different ways.
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**Definition**

A *globular set* $\mathcal{G}$ consists of sets $G_n$ for each $n$ and maps $s_n, t_n : G_{n+1} \to G_n$ for each $n$ such that the following *globularity conditions* hold:

\[
s_n \circ s_{n+1} = s_n \circ t_{n+1} \\
t_n \circ s_{n+1} = t_n \circ t_{n+1}
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We will write \( f : x \to y \) for an object \( f \) in \( G_{n+1} \) with \( s(f) = x \) and \( t(f) = y \).
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Let the objects of the globular set be it’s 0-cells, morphisms between these be 1-cells, . . .
Examples

- Ordinary 1-categories
- 2-categories, 3-categories, ...
- Monoidal categories, braided monoidal categories, ...
- Martin Löf Type Theory/non-Cubical Homotopy Type Theory
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**Unbiased vs Biased**

In an *unbiased* definition, we allow every possible composition operation.

In a *biased* definition, we only allow a subset of these operations.
Composition in infinity categories

Composition of 2 cells

Composition along a 1-boundary:

\[
\begin{array}{c}
\bullet \\
\alpha \uparrow \\
\downarrow \\
\beta \uparrow \\
\bullet
\end{array}
\]
Composition of 2 cells

Composition along a 1-boundary: $\bullet \xrightarrow{\beta} \bullet \xrightarrow{\alpha} \bullet$

Codimension along a 0-boundary: $\bullet \xleftarrow{\alpha} \bullet \xleftarrow{\beta} \bullet$
Higher compositions and pasting diagrams

3-cell composition
Definition

A *Stable* binary composition is a composition of an $n$-cell $a$ and an $m$-cell $b$ along their $(\min(n, m) - 1)$-boundary. We write this composition $a \cdot_k b$ where $k$ is the boundary dimension.
Claim

A definition of $\infty$-categories which only allows stable compositions is valid and equivalent to a fully unbiased definition.
Whisker-based composition scheme

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A definition of $\infty$-categories which only allows stable compositions is valid and equivalent to a fully unbiased definition.

Theorem
Every pasting diagram can be realised as a tree of stable binary composites. Furthermore this realisation respects source and target maps.
Pasting diagrams “with a focus” are uniquely generated by the following rules.

- The singleton pasting diagram \( x \), is a pasting diagram with focus \( x \).
- If \( \Gamma \) is a pasting diagram with focus \( x \), then \( \Gamma, y, f : x \to y \) is a pasting diagram with focus \( f \).
- If \( \Gamma \) is a pasting diagram with focus \( f : x \to y \), then it is also a pasting diagram with focus \( y \).

A pasting diagram is a pasting diagram with a 0-dimensional focus.
Proof Sketch

Definition

If $a$ is a cell and $x$ is a variable, the *principle replacement* $a\langle x \rangle$ is given recursively by:

- If $a = b \cdot c$ then $a\langle x \rangle = b \cdot (c\langle x \rangle)$.
- If $a$ is a variable then $a\langle x \rangle = x$. 
Proof Sketch

Definition

If \( a \) is a cell and \( x \) is a variable, the *principle replacement* \( a(x) \) is given recursively by:

- If \( a = b \cdot c \) then \( a(x) = b \cdot (c(x)) \).
- If \( a \) is a variable then \( a(x) = x \).

Definition

For pasting diagram \( \Gamma \), define its *stabilised form* \( S(\Gamma) \) by induction:

- If \( \Gamma \) is a singleton \( x \), then \( S(\Gamma) = x \).
- If \( \Gamma = \Delta, y, f \) and \( \text{dim}(f) > \text{dim}(\Delta) \) then \( S(\Gamma) = S(\Delta)f \).
- If \( \Gamma = \Delta, y, f \) and \( \text{dim}(f) \leq \text{dim}(\Delta) \) then \( S(\Gamma) = S(\Delta) \cdot (\delta^+(\text{dim}(y), S(\Delta)))f \).
Example
Conclusions

- We introduced the notion of a stable composite.
- We define a translation $S$ from an arbitrary pasting diagram to a tree of stable binary composites.
- This translation $S$ respects boundaries.
- Future aims include:
  - Proving the existence of an equivalence.
  - Generalise the work to prove the viability of many composition schemes.