Coinductive Invertibility in Higher Categories

Alex Rice

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Outline

1. Higher Categories
2. Equality in Higher Categories
3. Forms of invertibility and results
What is higher category?

In a regular category there are:

- A collection of objects;
- Between each pair objects there is a collection of morphisms between them.
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- Between each pair objects there is a collection of morphisms between them.

In higher category theory, we study cases where there is more structure on the collections of morphisms.
In a 2-category, the collections of morphisms form categories.
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In a 2-category, the collections of morphisms form categories. In a 3-category, the collections of morphisms form 2-categories. In an $\omega$-category, this continues.
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**Definition**

A *globular set* $\mathcal{G}$ is a collection of objects $|\mathcal{G}|$ and for each $x, y \in \mathcal{G}$ a globular set $\mathcal{G}_{x,y}$.
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An object $f \in |\mathcal{G}_{x,y}|$ will be called a morphism between $x$ and $y$, and will be written $f : x \to y$. 
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Let the objects of the globular set be it’s 0-cells, morphisms between these be 1-cells, . . .
Composition in infinity categories

Composition of 1 cells

\[ x \xrightarrow{f} y \xrightarrow{g} z \text{ written } f \star_1 g. \]
Composition in infinity categories

Composition of 1 cells

\[ \begin{array}{ccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \end{array} \] written \( f \star_1 g \).

Composition of 2 cells

Codimension 1:

\[ \begin{array}{ccc} \bullet & \xrightarrow{\beta} & \bullet \\ \downarrow{\alpha} & & \downarrow{\beta} \end{array} \] written \( \alpha \star_1 \beta \).
Composition in infinity categories

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Composition of 2 cells

Codimension 1:

\[ \bullet \xrightarrow{\beta \uparrow} \bullet \text{ written } \alpha \star_1 \beta. \]

Codimension 2:

\[ \bullet \xrightarrow{\alpha \uparrow} \bullet \xrightarrow{\beta \uparrow} \bullet \text{ written } \alpha \star_2 \beta. \]
Infinity categories also have identity cells. For each $n$-cell $f$ there is an $(n + 1)$-cell $\text{id}_f : f \to f$. 
Infinity categories also have identity cells. For each $n$-cell $f$ there is an $(n+1)$-cell $\text{id}_f : f \to f$. Regular categories have associativity and unit laws. These are also present in $\omega$-categories.
Topological spaces are nice examples of $\omega$-categories. Take a topological space $X$. 
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- 0 cells: $X$
- 1 cells: Paths between points
- 2 cells: Homotopies between paths
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Identities are given by constant paths/homotopies. Composition is given by path composition.
Fundamental $\omega$-groupoid

Similar to the topological space example, any type $X$ forms an $\omega$-category.
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Identities given by reflexivity proofs. Composition is transitivity of equality.
\( \omega \text{-Cat} \)

\textbf{Cat}, the category of (small) categories, forms a 2-category with:

- 0-cells: Categories;
- 1-cells: Functors;
- 2-cells: Natural transformations.
$\omega$-$\text{Cat}$

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$\omega\text{-Cat}$

$\textbf{Cat}$, the category of (small) categories, forms a 2-category with:
- 0-cells: Categories;
- 1-cells: Functors;
- 2-cells: Natural transformations.

Similarly $\textbf{2-Cat}$, the category of 2-categories, forms a 3-category. The category of $\omega$-categories, $\omega\text{-Cat}$, is itself an $\omega$-category.
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Isomorphism

In categories talking about whether two objects are the same or whether an object is unique is often the incorrect perspective. Instead, it is usual to talk about two objects being isomorphic, or an object being unique up to isomorphism.

**Definition**

Objects $X$ and $Y$ are *isomorphic* if there are morphisms $f : X \to Y$ and $g : Y \to X$ with $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. 
When comparing categories, the notion of isomorphism is too restrictive. Instead the notion of equivalence is used.
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**Definition**

An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta : \text{id}_\mathcal{C} \simeq GF$ and $\epsilon : FG \simeq \text{id}_\mathcal{D}$.
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The need for equivalence here arises as $\textbf{Cat}$ is a 2-category.
When using $n$-categories for $n$ larger than 2 or $\omega$-categories, an equivalence also becomes too restrictive because of its use of natural isomorphisms.
When using $n$-categories for $n$ larger than 2 or $\omega$-categories, an equivalence also becomes too restrictive because of its use of natural isomorphisms. Ideally, a version of equivalence where the natural isomorphisms are themselves equivalences is required. This leads naturally to a coinductive definition.
Quasi-invertibility

Definition

Given an $n$-cell $f : x \to y$, a quasi-invertible structure on $f$ is a tuple $(f^{-1}, f_R, f_L, f_R I, f_L I)$ where:

- $f^{-1}$ is an $n$-cell $y \to x$;
- $f_R$ is an $(n + 1)$-cell $f \star_1 f^{-1} \to \text{id}_x$;
- $f_L$ is an $(n + 1)$-cell $f^{-1} \star_1 f \to \text{id}_y$.
- $f_R I$ is a quasi-invertible structure on $f_R$.
- $f_L I$ is a quasi-invertible structure on $f_L$. 

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Coinductive Invertibility in Higher Categories
Properties of higher categories

$\omega$-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each $n$-cell $f : x \to y$, there are $(n + 1)$-cells, known as unitors, $\lambda_f : \text{id}_x \star_1 f \to f$ and $\rho_f : f \star_1 \text{id}_y \to f$.

- Given $f, g, h$, $n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)$.

- For compatible morphisms $f, g, h, j$, we have an interchanger $i_{f,g,h,j} : (f \star_n g) \star_1 (h \star_n j) \to (f \star_1 h) \star_n (g \star_1 j)$.

- For suitable $f, g$ and $n > 1$, there is a cell $\text{id}_f \star_{n+1} \text{id}_g \to \text{id}_{f \star_ng}$.
Higher Categories
Equality in Higher Categories
Forms of invertibility and results

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It is required that all these morphisms have quasi-invertible structures.
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- For \(n > 0\) and each \(n\)-cell \(f : x \to y\), there are \((n+1)\)-cells, known as unitors, \(\lambda_f : \text{id}_x \ast_1 f \to f\) and \(\rho_f : f \ast_1 \text{id}_y \to f\).

- Given \(f, g, h, n > 1\)-cells with suitable composition defined, we have an associator \(a_{f,g,h} : (f \ast_1 g) \ast_1 h \to f \ast_1 (g \ast_1 h)\).

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- For suitable \(f, g\) and \(n > 1\), there is a cell \(\text{id}_f \ast_{n+1} \text{id}_g \to \text{id}_{f \ast_n g}\).

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Further it is required that the \(\omega\)-category “respects the graphical calculus”.
## String diagrams

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<thead>
<tr>
<th>Pasting diagrams</th>
<th>String diagrams</th>
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\[ \begin{align*}
g & \Rightarrow f \\
g^{-1} & \Rightarrow f^{-1}
\end{align*} \]
In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.
Respecting the graphical calculus

**Theorem**

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.

**Definition**

An $\omega$-category respects the graphical calculus if, for any pair of string diagrams with a planar isotopy between them, there is a quasi-invertible 3-cell from the cell represented by the first to the cell represented by the second.
Properties of quasi-invertible structures

- Given a quasi-invertible structure of $f$, there exists a quasi-invertible structure on $f^{-1}$.
- There is a quasi-invertible structure on any identity morphism.
Limitations of quasi-invertibility

Take the $\omega$-category generated by 0-cells $x$ and $y$ and a quasi-invertible morphism $f : x \rightarrow y$. 
Limitations of quasi-invertibility

Take the $\omega$-category generated by 0-cells $x$ and $y$ and a quasi-invertible morphism $f : x \to y$. 

\[
\begin{align*}
&\begin{array}{c}
f_R \downarrow \quad f \downarrow \quad f^{-1} \downarrow \quad f_L \downarrow \\
&\end{array}
\end{align*}
\]
Invertibility in type theory

Inverses of \( f : A \rightarrow B \):

- Quasi-invertible:

  \[ \text{qinv}(f) : \Sigma_{g : B \rightarrow A} f \circ g \sim \text{id}_B \times g \circ f \sim \text{id}_A \]

- Bi-invertible:

  \[ \text{binv}(f) : \text{linv}(f) \times \text{rinv}(f) \]
  \[ \text{linv}(f) : \Sigma_{g : B \rightarrow A} g \circ f \sim \text{id}_A \]
  \[ \text{rinv}(f) : \Sigma_{g : B \rightarrow A} f \circ f \sim \text{id}_B \]

- Half-adjoint invertible:

  \[ \text{ishai}(f) : \Sigma_{g : B \rightarrow A} \Sigma_{\eta : g \circ f \sim \text{id}_A} \Sigma_{\epsilon : f \circ g \sim \text{id}_B} \prod_{x : A} f(\eta x) = \epsilon(fx) \]
Given an $n$-cell $f : x \to y$, a bi-invertible structure on $f$ is a tuple $(f^*, *f, f_R, f_L, f_R BI, f_L BI)$ where:

- $f^*$ is an $n$-cell $y \to x$;
- $*f$ is an $n$-cell $y \to x$;
- $f_R$ is an $(n + 1)$-cell $f \star_1 f^* \to \text{id}_x$;
- $f_L$ is an $(n + 1)$-cell $*f \star_1 f \to \text{id}_y$.
- $f_R BI$ is a bi-invertible structure on $f_R$.
- $f_L BI$ is a bi-invertible structure on $f_L$. 
Properties of bi-invertible structures

- Any quasi-invertible structure can be converted to a bi-invertible structure.
- Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.
- Given a bi-invertible structure on $f, f, f^*, *f, \ldots$ there are bi-invertible structures on both $f^*$ and $*f$. 

These are proved using coinduction and the results have been formalised in Agda using sized types.
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These are proved using coinduction and the results have been formalised in Agda using sized types.
Half-adjoint invertibility

Definition

Given an $n$-cell $f : x \to y$, a half-adjoint invertible structure on $f$ is a tuple $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f^{HAI}, \beta_f^{HAI}, \gamma_f^{HAI})$ where:

- $f'$ is an $n$-cell $y \to x$;
- $\alpha_f$ is an $(n + 1)$-cell $f \star_1 f' \to \text{id}_y$;
- $\beta_f$ is an $(n + 1)$-cell $\text{id}_x \to f' \star_1 f$;
- $\gamma_f$ is an $(n + 2)$-cell $(\lambda_{f'}^{-1} \star_1 (\beta_f \star_2 \text{id}_{f'}) \star_1 a_{f', f, f'} \star_1 (\text{id}_{f'} \star_2 \alpha_f) \star_1 \rho_{f'}) \to \text{id}_{f'}$;
- $\alpha_f^{HAI}$ is a half-adjoint invertible structure on $\alpha_f$;
- $\beta_f^{HAI}$ is a half-adjoint invertible structure on $\beta_f$;
- $\gamma_f^{HAI}$ is a half-adjoint invertible structure on $\gamma_f$. 

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Half-adjoint invertibility

\[ f' \xrightarrow{\alpha f} f' \]

\[ \xrightarrow{\beta f} \]

\[ f' \xrightarrow{\gamma f} f' \]
A *adjoint equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence $(F, G, \eta, \epsilon)$ such that $F \dashv G$ with unit $\eta$ and counit $\epsilon$. An adjoint equivalence is precisely a half-adjoint invertible structure in $\text{Cat}$. 

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Coinductive Invertibility in Higher Categories
Definition

A *adjoint equivalence* between categories \( \mathcal{C} \) and \( \mathcal{D} \) is an equivalence \((F, G, \eta, \epsilon)\) such that \( F \dashv G \) with unit \( \eta \) and counit \( \epsilon \).

An adjoint equivalence is precisely a half-adjoint invertible structure in \( \text{Cat} \).
**Main theorem**

**Theorem**

Let $G$ be a globular set with the given higher category properties. Let $n > 0$ and $f$ be an $n$-cell of $G$. Then the following are equivalent:

- $f$ has a bi-invertible structure.
- $f$ has a quasi-invertible structure.
- $f$ has a half-adjoint invertible structure.
Further work

- A limitation of quasi-invertible structures was presented earlier. Do bi-invertible structures and half-adjoint invertible structures have the same limitation?

- The “respects the graphical calculus” condition is slightly mysterious. It would be good to find a set of more concrete conditions from which it follows.

- Can coinduction be used to nicely describe other parts of higher category theory?
A bi-invertible structure \((f^*, *f, f_R, f_L, \ldots)\) of a cell \(f\) induces a half-adjoint invertible structure \((f^*, f_R, \ldots)\) on \(f\).
Proof

Let \((f^*, f^f, f^R, f^L, f^R Bl, f^L Bl)\) be a bi-invertible structure on \(f\). Then we give the right-adjoint invertible structure 
\((f^*, f^R, \beta_f, \gamma_f, f^R HA\|, \beta_f HA\|, \gamma_f HA\|)\) where:
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\begin{align*}
\beta_f & \Rightarrow f^* \\
\gamma_f & \Rightarrow f^*
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\[
\beta_f
\]

\[
\gamma_f
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\beta_f & \quad \beta_f \\
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\[
\begin{align*}
\beta_f &= f^* \Rightarrow (f_R) \Rightarrow (f_L) \\
\gamma_f &= f^* \Rightarrow (f_L) \Rightarrow (f_R)
\end{align*}
\]