Coinductive Invertibility in Higher Categories

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Outline

1 Higher Categories

2 Equality in Higher Categories

3 Forms of invertibility and results
What is higher category?

In a regular category there are:

- A collection of objects;
- Between each pair objects there is a collection of morphisms between them.
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In a regular category there are:

- A collection of objects;
- Between each pair objects there is a collection of morphisms between them.

In higher category theory, we study cases where there is more structure on the collections of morphisms.
In a 2-category, the collections of morphisms form categories.
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In a 2-category, the collections of morphisms form categories. In a 3-category, the collections of morphisms form 2-categories. In an $\omega$-category, this continues.
Globular Sets

ω-categories can take the shape of globular sets.
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**Definition**

A *globular set* $\mathcal{G}$ is a collection of objects $|\mathcal{G}|$ and for each $x, y \in \mathcal{G}$ a globular set $\mathcal{G}_{x,y}$. 
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An object \( f \in |\mathcal{G}_{x,y}| \) will be called a morphism between \( x \) and \( y \), and will be written \( f : x \to y \).
Globular Sets

$\omega$-categories can take the shape of globular sets. These are usually defined as presheaves.

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Let the objects of the globular set be it’s 0-cells, morphisms between these be 1-cells, . . .
Composition in infinity categories

Composition of 1 cells

\[ x \xrightarrow{f} y \xrightarrow{g} z \quad \text{written } f \star_1 g. \]
Composition in infinity categories

Composition of 1 cells

\[ x \overset{f}{\to} y \overset{g}{\to} z \text{ written } f \star_1 g. \]

Composition of 2 cells

Codimension 1: \[ \bullet \overset{\alpha}{\uparrow} \overset{\beta}{\downarrow} \overset{}{\bullet} \text{ written } \alpha \star_1 \beta. \]
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Composition of 2 cells

Codimension 1: \[ \bullet \xrightarrow{\beta \uparrow} \bullet \text{ written } \alpha \star_1 \beta. \]

Codimension 2: \[ \bullet \xrightarrow{\alpha \uparrow} \bullet \xrightarrow{\beta \uparrow} \bullet \text{ written } \alpha \star_2 \beta. \]
Infinity categories also have identity cells.
For each $n$-cell $f$ there is an $(n+1)$-cell $\text{id}_f : f \to f$. 
Infinity categories also have identity cells. For each $n$-cell $f$ there is an $(n + 1)$-cell $\text{id}_f : f \to f$. Regular categories have associativity and unit laws. These are also present in $\omega$-categories.
Topological spaces are nice examples of $\omega$-categories. Take a topological space $X$. 

0 cells: $X$

1 cells: Paths between points

2 cells: Homotopies between paths

... 

Identities are given by constant paths/homotopies.
Composition is given by path composition.
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Identities are given by constant paths/homotopies. Composition is given by path composition.
Similar to the topological space example, any type $X$ forms an $\omega$-category.
Fundamental $\omega$-groupoid

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- 0 cells: terms of type $X$;
- Higher cells: terms of equality types.
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Identities given by reflexivity proofs.
Composition is transitivity of equality.
\textbf{\(\omega\text{-Cat}\)}, the category of (small) categories, forms a 2-category with:

- 0-cells: Categories;
- 1-cells: Functors;
- 2-cells: Natural transformations.
\[ \text{\(\omega\)-Cat} \]

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Similarly, \textbf{2-Cat}, the category of 2-categories, forms a 3-category.
\textbf{Cat}, the category of (small) categories, forms a 2-category with:

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Similarly \textbf{2-Cat}, the category of 2-categories, forms a 3-category.

The category of \(\omega\)-categories, \(\omega\text{-Cat}\), is itself an \(\omega\)-category.
Isomorphism

In categories talking about whether two objects are the same or whether an object is unique is often the incorrect perspective.
In categories talking about whether two objects are the same or whether an object is unique is often the incorrect perspective. Instead, it is usual to talk about two objects being isomorphic, or an object being unique up to isomorphism.

**Definition**

Objects $X$ and $Y$ are *isomorphic* if there are morphisms $f : X \to Y$ and $g : Y \to X$ with $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. 

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Coinductive Invertibility in Higher Categories
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An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms

$\eta : \text{id}_\mathcal{C} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \text{id}_\mathcal{D}$
When comparing categories, the notion of isomorphism is too restrictive. Instead the notion of equivalence is used.

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The need for equivalence here arises as $\textbf{Cat}$ is a 2-category.
When using $n$-categories for $n$ larger than 2 or $\omega$-categories, an equivalence also becomes too restrictive because of its use of natural isomorphisms.
When using $n$-categories for $n$ larger than 2 or $\omega$-categories, an equivalence also becomes too restrictive because of its use of natural isomorphisms. Ideally, a version of equivalence where the natural isomorphisms are themselves equivalences is required. This leads naturally to a coinductive definition.
Quasi-invertibility

Definition

Given an $n$-cell $f : x \to y$, a quasi-invertible structure on $f$ is a tuple $(f^{-1}, f_R, f_L, f_R I, f_L I)$ where:

- $f^{-1}$ is an $n$-cell $y \to x$;
- $f_R$ is an $(n+1)$-cell $f \star_1 f^{-1} \to \text{id}_x$;
- $f_L$ is an $(n+1)$-cell $f^{-1} \star_1 f \to \text{id}_y$.
- $f_R I$ is a quasi-invertible structure on $f_R$.
- $f_L I$ is a quasi-invertible structure on $f_L$. 
Properties of higher categories

$\omega$-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each $n$-cell $f : x \to y$, there are $(n + 1)$-cells, known as unitors, $\lambda_f : \text{id}_x \star_1 f \to f$ and $\rho_f : f \star_1 \text{id}_y \to f$.

- Given $f, g, h, n > 1$-cells with suitable composition defined, we have an associator $a_{f, g, h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)$.

- For compatible morphisms $f, g, h, j$, we have an interchanger $i_{f, g, h, j} : (f \star_n g) \star_1 (h \star_n j) \to (f \star_1 h) \star_n (g \star_1 j)$.

- For suitable $f, g$ and $n > 1$, there is a cell $\text{id}_f \star_{n+1} \text{id}_g \to \text{id}_{f \star_n g}$. 
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ω-categories have coherence properties. Here we specify a minimal set of properties required:

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It is required that all these morphisms have quasi-invertible structures.
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- For \(n > 0\) and each \(n\)-cell \(f : x \to y\), there are \((n + 1)\)-cells, known as unitors, \(\lambda_f : \text{id}_x \star_1 f \to f\) and \(\rho_f : f \star_1 \text{id}_y \to f\).

- Given \(f, g, h, n > 1\)-cells with suitable composition defined, we have an associator \(a_{f,g,h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)\).

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- For suitable \(f, g\) and \(n > 1\), there is a cell \(\text{id}_f \star_{n+1} \text{id}_g \to \text{id}_{f \star ng}\).

It is required that all these morphisms have quasi-invertible structures.

Further it is required that the \(\omega\)-category “respects the graphical calculus”.

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Coinductive Invertibility in Higher Categories
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<th>String diagrams</th>
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\[ g_R \xrightarrow{f_R} g^{-1} f^{-1} \]
Respecting the graphical calculus

Theorem

*In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.*
Theorem

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.

Definition

An ω-category respects the graphical calculus if, for any pair of string diagrams with a planar isotopy between them, there is a quasi-invertible 3-cell from the cell represented by the first to the cell represented by the second.
Properties of quasi-invertible structures

- Given a quasi-invertible structure of $f$, there exists a quasi-invertible structure on $f^{-1}$.
- There is a quasi-invertible structure on any identity morphism.
Limitations of quasi-invertibility

Take the $\omega$-category generated by 0-cells $x$ and $y$ and a quasi-invertible morphism $f : x \to y$. 
Limitations of quasi-invertibility

Take the \( \omega \)-category generated by 0-cells \( x \) and \( y \) and a quasi-invertible morphism \( f : x \to y \).
Invertibility in type theory

Inverses of $f : A \to B$:

- **Quasi-invertible:**
  
  $$\text{qinv}(f) : \Sigma_{g : B \to A} f \circ g \sim \text{id}_B \times g \circ f \sim \text{id}_A$$

- **Bi-invertible:**
  
  $$\text{binv}(f) : \text{linv}(f) \times \text{rinv}(f)$$
  
  $$\text{linv}(f) : \Sigma_{g : B \to A} g \circ f \sim \text{id}_A$$
  
  $$\text{rinv}(f) : \Sigma_{g : B \to A} f \circ f \sim \text{id}_B$$

- **Half-adjoint invertible:**
  
  $$\text{ishai}(f) : \Sigma_{g : B \to A} \Sigma_{\eta : g \circ f \sim \text{id}_A} \Sigma_{\epsilon : f \circ g \sim \text{id}_B} \prod_{x : A} f(\eta x) = \epsilon(fx)$$
Bi-invertibility

Given an $n$-cell $f : x \to y$, a bi-invertible structure on $f$ is a tuple $(f^*, *f, f_R, f_L, f_R BI, f_L BI)$ where:

- $f^*$ is an $n$-cell $y \to x$;
- $*f$ is an $n$-cell $y \to x$;
- $f_R$ is an $(n+1)$-cell $f \ast_1 f^* \to \text{id}_x$;
- $f_L$ is an $(n+1)$-cell $*f \ast_1 f \to \text{id}_y$.
- $f_R BI$ is a bi-invertible structure on $f_R$.
- $f_L BI$ is a bi-invertible structure on $f_L$. 

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Any quasi-invertible structure can be converted to a bi-invertible structure.

Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.

Given a bi-invertible structure on $f, f, f^*, *f, \ldots$ there are bi-invertible structures on both $f^*$ and $*f$. 
Any quasi-invertible structure can be converted to a bi-invertible structure.

Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.

Given a bi-invertible structure on $f, f, f^*, *f, \ldots$ there are bi-invertible structures on both $f^*$ and $*f$.

These are proved using coinduction and the results have been formalised in Agda using sized types.
Half-adjoint invertibility

Definition

Given an $n$-cell $f : x \to y$, a half-adjoint invertible structure on $f$ is a tuple $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f HAI, \beta_f HAI, \gamma_f HAI)$ where:

- $f'$ is an $n$-cell $y \to x$;
- $\alpha_f$ is an $(n + 1)$-cell $f \star_1 f' \to \text{id}_y$;
- $\beta_f$ is an $(n + 1)$-cell $\text{id}_x \to f' \star_1 f$;
- $\gamma_f$ is an $(n + 2)$-cell
  \[
  (\lambda_{f'}^{-1} \star_1 (\beta_f \star_2 \text{id}_{f'}) \star_1 a_{f',f,f'} \star_1 (\text{id}_{f'} \star_2 \alpha_f) \star_1 \rho_{f'}) \to \text{id}_{f'};
  \]
- $\alpha_f HAI$ is a half-adjoint invertible structure on $\alpha_f$;
- $\beta_f HAI$ is a half-adjoint invertible structure on $\beta_f$;
- $\gamma_f HAI$ is a half-adjoint invertible structure on $\gamma_f$. 
Half-adjoint invertibility

\[ f' \xrightarrow{\beta_f} \alpha_f \xrightarrow{\gamma_f} f' \]
**Definition**

A *adjoint equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence $(F, G, \eta, \epsilon)$ such that $F \dashv G$ with unit $\eta$ and counit $\epsilon$. An adjoint equivalence is precisely a half-adjoint invertible structure in $\text{Cat}$. 

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Definition

An adjoint equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence $(F, G, \eta, \epsilon)$ such that $F \dashv G$ with unit $\eta$ and counit $\epsilon$.

An adjoint equivalence is precisely a half-adjoint invertible structure in $\textbf{Cat}$. 
Main theorem

Theorem

Let $G$ be a globular set with the given higher category properties. Let $n > 0$ and $f$ be an $n$-cell of $G$. Then the following are equivalent:

- $f$ has a bi-invertible structure.
- $f$ has a quasi-invertible structure.
- $f$ has a half-adjoint invertible structure.
Further work

- A limitation of quasi-invertible structures was presented earlier. Do bi-invertible structures and half-adjoint invertible structures have the same limitation?
- The “respects the graphical calculus” condition is slightly mysterious. It would be good to find a set of more concrete conditions from which it follows.
- Can coinduction be used to nicely describe other parts of higher category theory?
Theorem

A bi-invertible structure \((f^*, \ast f, f_R, f_L, \ldots)\) of a cell \(f\) induces a half-adjoint invertible structure \((f^*, f_R, \ldots)\) on \(f\).
Proof

Let \((f^*, *f, f_R, f_L, f_R BI, f_L BI)\) be a bi-invertible structure on \(f\). Then we give the right-adjoint invertible structure

\((f^*, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)\) where:
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\[
\beta_f
\]

\[
\begin{array}{c}
\overset{f^*}{f} \\
\overset{f}{f_R} \\
\overset{f_R}{f_L} \\
\overset{\gamma_f}{f_L}
\end{array}
\]
Let \((f^*, *f, f_R, f_L, f_RBI, f_LBI)\) be a bi-invertible structure on \(f\). Then we give the right-adjoint invertible structure \((f^*, f_R, \beta_f, \gamma_f, f_RHAI, \beta_fHAI, \gamma_fHAI)\) where:

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\begin{align*}
\beta_f & : f^* \to f \\
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