Coinductive Invertibility in Higher Categories

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Outline

What is higher category?

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In higher category theory, we study cases where there is more structure on the collections of morphisms.

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In an ω-category, this continues.

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Let the objects of the globular set be it's 0-cells, morphisms between these be 1-cells, ...

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Codimension 2:
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written $\alpha \star_2 \beta$.

Coherence for infinity categories

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Regular categories have associativity and unit laws. These are also present in ω-categories.

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Composition is given by path composition.

Fundamental ω-groupoid

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Identities given by reflexivity proofs.
Composition is transitivity of equality.

ω-**Cat**

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Similarly **2-Cat**, the category of 2-categories, forms a 3-category.
The category of ω-categories, ω-**Cat**, is itself an ω-category.

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Definition

Objects X and Y are *isomorphic* if there are morphisms $f : X \to Y$ and $g : Y \to X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

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Definition

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The need for equivalence here arises as **Cat** is a 2-category.

Equivalence in higher categories

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Ideally, a version of equivalence where the natural isomorphisms are themselves equivalences is required. This leads naturally to a coinductive definition.

Quasi-invertibility

Definition

Given an n-cell $f : x \to y$, a *quasi-invertible* structure on f is a tuple $(f^{-1}, f_R, f_L, f_R I, f_L I)$ where:

- f^{-1} is an n-cell $y \to x$;
- f_R is an $(n+1)$-cell $f \star_1 f^{-1} \to \mathrm{id}_x$;
- f_L is an $(n+1)$-cell $f^{-1} \star_1 f \to \mathrm{id}_y$.
- $f_R I$ is a quasi-invertible structure on f_R.
- $f_L I$ is a quasi-invertible structure on f_L.

Properties of higher categories

ω-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each n-cell $f : x \to y$, there are $(n+1)$-cells, known as unitors, $\lambda_f : \mathrm{id}_x \star_1 f \to f$ and $\rho_f : f \star_1 \mathrm{id}_y \to f$.
- Given f, g, h, $n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)$.
- For compatible morphisms f, g, h, j, we have an interchanger $i_{f,g,h,j} : (f \star_n g) \star_1 (h \star_n j) \to (f \star_1 h) \star_n (g \star_1 j)$.
- For suitable f, g and $n > 1$, there is a cell $\mathrm{id}_f \star_{n+1} \mathrm{id}_g \to \mathrm{id}_{f \star_n g}$.

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It is required that all these morphisms have quasi-invertible structures.
Further it is required that the ω-category "respects the graphical calculus".

String diagrams

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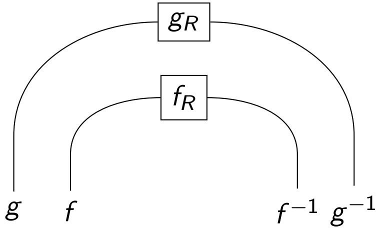

Respecting the graphical calculus

Theorem

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.

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Definition

An ω-category *respects the graphical calculus* if, for any pair of string diagrams with a planar isotopy between them, there is a quasi-invertible 3-cell from the cell represented by the first to the cell represented by the second.

Properties of quasi-invertible structures

- Given a quasi-invertible structure of f, there exists a quasi-invertible structure on f^{-1}.
- There is a quasi-invertible structure on any identity morphism.

Limitations of quasi-invertibility

Take the ω-category generated by 0-cells x and y and a quasi-invertible morphism $f : x \to y$.

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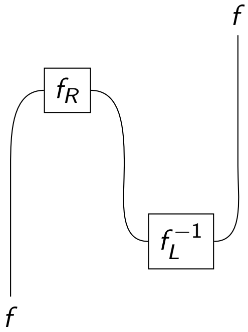

Invertibility in type theory

Inverses of $f : A \to B$:

- Quasi-invertible:

$$\mathrm{qinv}(f) : \Sigma_{g:B\to A}\, f \circ g \sim \mathrm{id}_B \times g \circ f \sim \mathrm{id}_A$$

- Bi-invertible:

$$\begin{aligned}\mathrm{binv}(f) &: \mathrm{linv}(f) \times \mathrm{rinv}(f)\\ \mathrm{linv}(f) &: \Sigma_{g:B\to A}\, g \circ f \sim \mathrm{id}_A\\ \mathrm{rinv}(f) &: \Sigma_{g:B\to A}\, f \circ f \sim \mathrm{id}_B\end{aligned}$$

- Half-adjoint invertible:

$$\mathrm{ishai}(f) : \Sigma_{g:B\to A}\, \Sigma_{\eta:g\circ f\sim \mathrm{id}_A}\, \Sigma_{\epsilon:f\circ g\sim \mathrm{id}_B}\, \Pi_{x:A}\, f(\eta x) = \epsilon(fx)$$

Bi-invertibility

Given an n-cell $f : x \to y$, a *bi-invertible* structure on f is a tuple $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ where:

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- ${}^\star f$ is an n-cell $y \to x$;
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- $f_R BI$ is a bi-invertible structure on f_R.
- $f_L BI$ is a bi-invertible structure on f_L.

Properties of bi-invertible structures

- Any quasi-invertible structure can be converted to a bi-invertible structure.
- Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.
- Given a bi-invertible structure on f, $f, f^\star, {}^\star\!f, \dots$ there are bi-invertible structures on both $f^\star$ and ${}^\star\!f$.

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These are proved using coinduction and the results have been formalised in Agda using sized types.

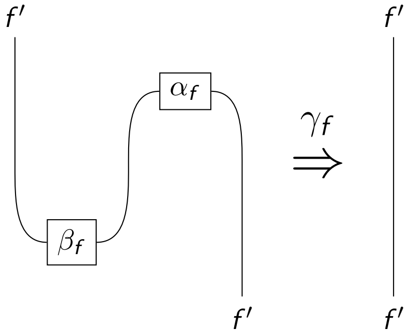
Half-adjoint invertibility

Definition

Given an n-cell $f : x \to y$, a *half-adjoint invertible* structure on f is a tuple $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f HAI, \beta_f HAI, \gamma_f HAI)$ where:

- f' is an n-cell $y \to x$;
- α_f is an $(n+1)$-cell $f \star_1 f' \to \mathrm{id}_y$;
- β_f is an $(n+1)$-cell $\mathrm{id}_x \to f' \star_1 f$;
- γ_f is an $(n+2)$-cell $(\lambda_{f'}^{-1} \star_1 (\beta_f \star_2 \mathrm{id}_{f'}) \star_1 a_{f',f,f'} \star_1 (\mathrm{id}_{f'} \star_2 \alpha_f) \star_1 \rho_{f'}) \to \mathrm{id}_{f'}$;
- $\alpha_f HAI$ is a half-adjoint invertible structure on α_f;
- $\beta_f HAI$ is a half-adjoint invertible structure on β_f;
- $\gamma_f HAI$ is a half-adjoint invertible structure on γ_f.

Half-adjoint invertibility



Adjoint equivalence

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An adjoint equivalence is precisely a half-adjoint invertible structure in **Cat**.

Main theorem

Theorem

Let G be a globular set with the given higher category properties. Let $n > 0$ and f be an n-cell of G. Then the following are equivalent:

- *f has a bi-invertible structure.*
- *f has a quasi-invertible structure.*
- *f has a half-adjoint invertible structure.*

Further work

- A limitation of quasi-invertible structures was presented earlier. Do bi-invertible structures and half-adjoint invertible structures have the same limitation?
- The "respects the graphical calculus" condition is slightly mysterious. It would be good to find a set of more concrete conditions from which it follows.
- Can coinduction be used to nicely describe other parts of higher category theory?

Bi-invertibility implies half-adjoint invertibility

Theorem

A bi-invertible structure $(f^\star, {}^\star f, f_R, f_L, \ldots)$ *of a cell* f *induces a half-adjoint invertible structure* $(f^\star, f_R, \ldots)$ *on* f.

Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

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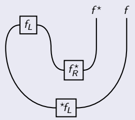

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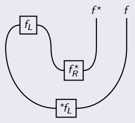

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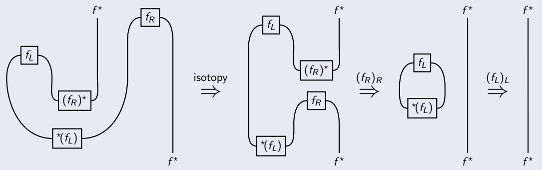

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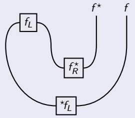

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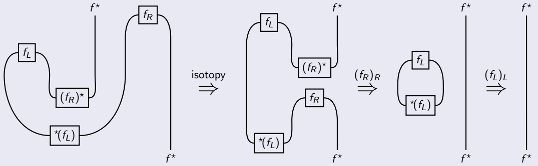

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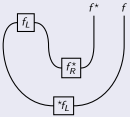

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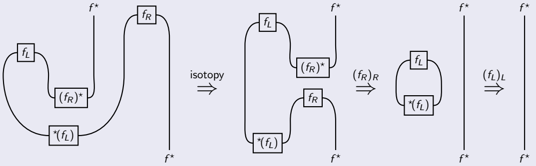

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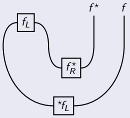

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