Coinductive Invertibility in Higher Categories

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Outline

1. Higher Categories
2. Equality in Higher Categories
3. Forms of invertibility and results
What is higher category?

In a regular category there are:

- A collection of objects;
- Between each pair objects there is a collection of morphisms between them.
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In a regular category there are:
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- Between each pair objects there is a collection of morphisms between them.

In higher category theory, we study cases where there is more structure on the collections of morphisms.
In a 2-category, the collections of morphisms form categories.
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In a 3-category, the collections of morphisms form 2-categories.
In an $\omega$-category, this continues.
ω-categories can take the shape of globular sets.
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**Definition**

A *globular set* $\mathcal{G}$ is a collection of objects $|\mathcal{G}|$ and for each $x, y \in \mathcal{G}$ a globular set $\mathcal{G}_{x,y}$. 
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An object $f \in |\mathcal{G}_{x,y}|$ will be called a morphism between $x$ and $y$, and will be written $f : x \to y$. 
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Let the objects of the globular set be its 0-cells, morphisms between these be 1-cells, ...
Composition in infinity categories

Composition of 1 cells

\[ x \xrightarrow{f} y \xrightarrow{g} z \] written \( f \star_1 g \).
Composition in infinity categories

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\[ x \xrightarrow{f} y \xrightarrow{g} z \] written \( f \star_1 g \).

Composition of 2 cells

Codimension 1:

\[ \bullet \xrightarrow{\beta \uparrow} \bullet \] written \( \alpha \star_1 \beta \).
Composition in infinity categories

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**Composition of 2 cells**

**Codimension 1:**

\[ \begin{array}{c}
\alpha \\
\beta
\end{array} \quad \text{written } \alpha \star_1 \beta. \]

**Codimension 2:**

\[ \begin{array}{c}
\alpha \\
\beta
\end{array} \quad \text{written } \alpha \star_2 \beta. \]
Infinity categories also have identity cells.
For each $n$-cell $f$ there is an $(n + 1)$-cell $\text{id}_f : f \to f$. 
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Topological spaces are nice examples of $\omega$-categories. Take a topological space $X$. 
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- 2 cells: Homotopies between paths
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Identities are given by constant paths/homotopies. Composition is given by path composition.
Similar to the topological space example, any type $X$ forms an $\omega$-category.
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- 0 cells: terms of type $X$;
- Higher cells: terms of equality types.
Fundamental $\omega$-groupoid

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Identities given by reflexivity proofs.
Composition is transitivity of equality.
$\omega$-Cat

**Cat**, the category of (small) categories, forms a 2-category with:

- 0-cells: Categories;
- 1-cells: Functors;
- 2-cells: Natural transformations.

Similarly, $\mathbf{2}$-Cat, the category of 2-categories, forms a 3-category. The category of $\omega$-categories, $\omega$-$\mathbf{Cat}$, is itself an $\omega$-category.
\[ \text{\( \omega \)-Cat} \]

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Isomorphism

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**Definition**

Objects \( X \) and \( Y \) are *isomorphic* if there are morphisms \( f : X \to Y \) and \( g : Y \to X \) with \( f \circ g = \text{id}_Y \) and \( g \circ f = \text{id}_X \).
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**Definition**

An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms

$\eta : \text{id}_\mathcal{C} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \text{id}_\mathcal{D}$
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The need for equivalence here arises as \( \textbf{Cat} \) is a 2-category.
When using $n$-categories for $n$ larger than 2 or $\omega$-categories, an equivalence also becomes too restrictive because of its use of natural isomorphisms.
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Given an $n$-cell $f : x \rightarrow y$, a quasi-invertible structure on $f$ is a tuple $(f^{-1}, f_R, f_L, f_R I, f_L I)$ where:

- $f^{-1}$ is an $n$-cell $y \rightarrow x$;
- $f_R$ is an $(n + 1)$-cell $f \star_1 f^{-1} \rightarrow \text{id}_x$;
- $f_L$ is an $(n + 1)$-cell $f^{-1} \star_1 f \rightarrow \text{id}_y$.
- $f_R I$ is a quasi-invertible structure on $f_R$.
- $f_L I$ is a quasi-invertible structure on $f_L$. 

Quasi-invertibility
Properties of higher categories

$\omega$-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each $n$-cell $f : x \to y$, there are $(n + 1)$-cells, known as unitors, $\lambda_f : \text{id}_x \ast_1 f \to f$ and $\rho_f : f \ast_1 \text{id}_y \to f$.
- Given $f, g, h, n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h} : (f \ast_1 g) \ast_1 h \to f \ast_1 (g \ast_1 h)$.
- For compatible morphisms $f, g, h, j$, we have an interchanger $i_{f,g,h,j} : (f \ast_n g) \ast_1 (h \ast_n j) \to (f \ast_1 h) \ast_n (g \ast_1 j)$.
- For suitable $f, g$ and $n > 1$, there is a cell $\text{id}_f \ast_{n+1} \text{id}_g \to \text{id}_{f \ast_n g}$.
Higher Categories
Equality in Higher Categories
Forms of invertibility and results

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It is required that all these morphisms have quasi-invertible structures.
Further it is required that the $\omega$-category “respects the graphical calculus”.
## String diagrams

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<th>Pasting diagrams</th>
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\[
g R \\
\]

\[
f R \\
\]

\[
g, f, f^{-1}, g^{-1} \\
\]
Respecting the graphical calculus

Theorem

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.
Respecting the graphical calculus

**Theorem**

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.

**Definition**

An $\omega$-category respects the graphical calculus if, for any pair of string diagrams with a planar isotopy between them, there is a quasi-invertible 3-cell from the cell represented by the first to the cell represented by the second.
Properties of quasi-invertible structures

- Given a quasi-invertible structure of $f$, there exists a quasi-invertible structure on $f^{-1}$.
- There is a quasi-invertible structure on any identity morphism.
Take the $\omega$-category generated by 0-cells $x$ and $y$ and a quasi-invertible morphism $f : x \to y$. 
Limitations of quasi-invertibility

Take the $\omega$-category generated by 0-cells $x$ and $y$ and a quasi-invertible morphism $f : x \to y$. 
Invertibility in type theory

Inverses of \( f : A \rightarrow B \):

- Quasi-invertible:
  \[
  \text{qinv}(f) : \Sigma_{g:A \rightarrow B} f \circ g \sim \text{id}_B \times g \circ f \sim \text{id}_A
  \]

- Bi-invertible:
  \[
  \begin{align*}
  \text{binv}(f) : & \text{linv}(f) \times \text{rinv}(f) \\
  \text{linv}(f) : & \Sigma_{g:A \rightarrow B} g \circ f \sim \text{id}_A \\
  \text{rinv}(f) : & \Sigma_{g:A \rightarrow B} f \circ f \sim \text{id}_B
  \end{align*}
  \]

- Half-adjoint invertible:
  \[
  \text{ishai}(f) : \Sigma_{g:A \rightarrow B} \Sigma_{\eta:C \rightarrow A} \Sigma_{\epsilon:C \rightarrow B} \prod_x A(f(\eta x)) = \epsilon(fx)
  \]
Given an $n$-cell $f : x \to y$, a bi-invertible structure on $f$ is a tuple $(f^*, \ast f, f_R, f_L, f_R BI, f_L BI)$ where:

- $f^*$ is an $n$-cell $y \to x$;
- $\ast f$ is an $n$-cell $y \to x$;
- $f_R$ is an $(n + 1)$-cell $f \ast_1 f^* \to \text{id}_x$;
- $f_L$ is an $(n + 1)$-cell $\ast f \ast_1 f \to \text{id}_y$.
- $f_R BI$ is a bi-invertible structure on $f_R$.
- $f_L BI$ is a bi-invertible structure on $f_L$. 
Properties of bi-invertible structures

- Any quasi-invertible structure can be converted to a bi-invertible structure.
- Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.
- Given a bi-invertible structure on \( f, f, f^*, *f, \ldots \) there are bi-invertible structures on both \( f^* \) and \( *f \).
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These are proved using coinduction and the results have been formalised in Agda using sized types.
**Definition**

Given an $n$-cell $f : x \to y$, a *half-adjoint invertible* structure on $f$ is a tuple $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f HAI, \beta_f HAI, \gamma_f HAI)$ where:

- $f'$ is an $n$-cell $y \to x$;
- $\alpha_f$ is an $(n + 1)$-cell $f \star_1 f' \to \text{id}_y$;
- $\beta_f$ is an $(n + 1)$-cell $\text{id}_x \to f' \star_1 f$;
- $\gamma_f$ is an $(n + 2)$-cell
  $$(\lambda_{f'}^{-1} \star_1 (\beta_f \star_2 \text{id}_{f'}) \star_1 a_{f',f',f'} \star_1 (\text{id}_{f'} \star_2 \alpha_f) \star_1 \rho_{f'}) \to \text{id}_{f'};$$
- $\alpha_f HAI$ is a half-adjoint invertible structure on $\alpha_f$;
- $\beta_f HAI$ is a half-adjoint invertible structure on $\beta_f$;
- $\gamma_f HAI$ is a half-adjoint invertible structure on $\gamma_f$. 

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Coinductive Invertibility in Higher Categories
Half-adjoint invertibility

\[ f' \xrightarrow{\alpha_f} f' \xrightarrow{\beta_f} f' \]

\[ \implies \gamma_f \]
Adjoint equivalence

Definition

A *adjoint equivalence* between categories \( \mathcal{C} \) and \( \mathcal{D} \) is an equivalence \((F, G, \eta, \epsilon)\) such that \( F \dashv G \) with unit \( \eta \) and counit \( \epsilon \).
Adjoint equivalence

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A adjoint equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence $(F, G, \eta, \epsilon)$ such that $F \dashv G$ with unit $\eta$ and counit $\epsilon$

An adjoint equivalence is precisely a half-adjoint invertible structure in $\textbf{Cat}$. 
Main theorem

Theorem

Let $G$ be a globular set with the given higher category properties. Let $n > 0$ and $f$ be an $n$-cell of $G$. Then the following are equivalent:

- $f$ has a bi-invertible structure.
- $f$ has a quasi-invertible structure.
- $f$ has a half-adjoint invertible structure.
Further work

- A limitation of quasi-invertible structures was presented earlier. Do bi-invertible structures and half-adjoint invertible structures have the same limitation?
- The “respects the graphical calculus” condition is slightly mysterious. It would be good to find a set of more concrete conditions from which it follows.
- Can coinduction be used to nicely describe other parts of higher category theory?
**Theorem**

A bi-invertible structure \((f^*, *f, f_R, f_L, \ldots)\) of a cell \(f\) induces a half-adjoint invertible structure \((f^*, f_R, \ldots)\) on \(f\).
Proof

Let \((f^*, f, f_R, f_L, f_R BI, f_L BI)\) be a bi-invertible structure on \(f\). Then we give the right-adjoint invertible structure 
\((f^*, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)\) where:
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\((f^*, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)\) where:

- \(\beta_f\) and \(\gamma_f\) are the coherence maps ensuring the invertibility conditions are satisfied.

![Diagram of \(\beta_f\)]
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\[
\begin{align*}
\beta_f & \quad \gamma_f \\
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