A Type Theoretic Approach to Semistrict Higher Categories

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12th May 2022
Outline

1. Globular Infinity Categories
2. Weak Infinity Categories
3. Semistrict infinity categories
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\[
\begin{array}{c}
\alpha \\
\downarrow \quad \quad \downarrow \\
\phantom{g} && \phantom{f}
\end{array}
\]
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\[ x \xymatrix{ & \alpha \ar[ur]^g \ar[dr]_f & y } \]

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\[ \begin{array}{ccc}
  x & \alpha & y \\
  f & \g & \\
\end{array} \]

... 

**Definition**

A *Globular Set* is a set $G$ with a globular set $G_{x,y}$ for each pair of objects $x, y \in G$. 

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Composition in Globular Sets

Composition of 1 cells

\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \]
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Composition along a 1-boundary:

\[ \bullet \xrightarrow{\beta \uparrow} \bullet \]

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Composition along a 0-boundary:

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In a strict infinity category we have binary composition of $n$-cells for along a $k$ boundary for all $k < n$.

**Composition**

If $f$ and $g$ are $n$-cells with the $k$-target of $f$ equalling the $k$-source of $g$ then there is an $n$-cell $f \circ_k g$. 
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**Composition**

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**Identities**

For each $n$-cell $f$ there is an $(n + 1)$-cell $\text{id}_f : f \to f$. 
If $0 \leq k < n$ and $f$, $g$, and $h$ are $n$-cells then:

$$f \circ_k (g \circ_k h) = (f \circ_k g) \circ_k h$$
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**Associativity of 1-cells**

Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ we have:

$$f \circ_0 (g \circ_0 h) = (f \circ_0 g) \circ_0 h$$
If $0 \leq k < n$ and $f$ is an $n$-cell with $k$-source $x$ and $k$-target $y$ then:

$$\text{id}^{n-k}(x) \circ_k f = f = f \circ_k \text{id}^{n-k}(y)$$
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Identity on 2-cell

Given $f, g : x \to y$ and $\alpha : f \to g$ we have:

\[
\begin{array}{cccccc}
\text{id}(x) & \alpha & \text{id}(\text{id}(x)) & \text{id}(x) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{id}(f) & f & \text{id}(x) & g \\
\end{array}
\]

\[
\begin{array}{cccccc}
\alpha & \downarrow & \text{id}(\text{id}(x)) & \downarrow & \text{id}(x) \\
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\alpha & \downarrow & \text{id}(\text{id}(x)) & \downarrow & \text{id}(x) \\
\text{id}(f) & f & \text{id}(x) & g \\
\end{array}
\]
If $0 \leq q < p < n$ and $a, b, c, d$ are $n$-cells then:

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$
If $0 \leq q < p < n$ and $a, b, c, d$ are $n$-cells then:

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$

Further if $f \circ_k g$ is well defined then:

$$\text{id}_f \circ_k \text{id}(g) = \text{id}(f \circ_k g)$$
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**Definition (Monoidal category)**

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A strict infinity category with one object and no non-identity $n$-cells for $n$ higher than 2 is a strict monoidal category.
If a category has all products and a terminal object, then it can be given the structure of a monoidal category.
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The monoidal product in **Set** is *not* strict.
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- For a 1-cell $f : x \to y$, there are unitors $\lambda_f : \text{id}_x \circ f \to f$ and $\rho_f : f \circ \text{id}_y$.
- $\lambda_{\text{id}_x}$ and $\rho_{\text{id}_x}$ are both arrows $\text{id}_x \circ \text{id}_x \to \text{id}_x$. We can ask that they be isomorphic.
- This isomorphism will also be subject to coherence conditions.
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For a 1-cell \( f : x \to y \), there are unitors \( \lambda_f : \text{id}_x \circ f \to f \) and \( \rho_f : f \circ \text{id}_y \). \( \lambda_{id_x} \) and \( \rho_{id_x} \) are both arrows \( \text{id}_x \circ \text{id}_x \to \text{id}_x \). We can ask that they be isomorphic. This isomorphism will also be subject to coherence conditions.

It quickly becomes apparent that we need a more uniform way to package this coherence data.
A pasting diagram represents a composition that can be done in an infinity category.
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The compositions we have already seen form pasting diagrams.

We can also form more complicated compositions as pasting diagrams.
Pasting diagrams for 1-categories are simply chains of 1-cells:
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\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{\cdots} \bullet \]

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\[
\bullet 
\xrightarrow{f} 
\bullet 
\xrightarrow{g} 
\bullet 
\xrightarrow{} 
\bullet 
\xrightarrow{} 
\cdot\cdot\cdot 
\xrightarrow{} 
\bullet
\]

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- In a strict infinity category, every (higher dimensional) pasting diagram has exactly one composite.
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In a strict infinity category, every (higher dimensional) pasting diagram has exactly one composite.
For weak infinity categories, we weaken the exactness condition to uniqueness up to isomorphism.
Motto for weak infinity categories

- Every pasting diagram has a composite
- Given 2 parallel arrows $s, t$ generated from the whole pasting diagram, there is a higher dimensional arrow from $s$ to $t$. 

Taking the composite of the diagram:

\[
\begin{array}{c}
\bullet \\
\downarrow f \quad \downarrow \quad \downarrow \quad \downarrow \\
\bullet \\
\downarrow g \\
\bullet
\end{array}
\]

This gives the composite $f \circ g$.

Over the singleton pasting diagram $x$ and taking $s = x$ and $t = x$ we get a term from $x$ to $x$ representing the identity on $x$. 


Weak Composition

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\bullet \xrightarrow{x}
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CaTT is a type theory for weak infinity categories. It allows us to build and describe the operations of an infinity category.
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- **Types**: A type contains all the information of the *sources* and *targets* for a term.
- **Substitutions**: A substitution is a *morphism* between contexts.
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Types have 2 constructors, the star constructor and the arrow constructor.

- If a term is a 0-cell in our infinity category, then it has type \(\star\).
- Otherwise a term is an \((n + 1)\)-cell between parallel \(n\)-cells \(f\) and \(g\), in which case it has type:

\[
f \rightarrow_A g
\]

where \(A\) is the (common) type of \(f\) and \(g\).
The crucial part of CaTT is the Coh constructor, which captures the motto for weak composition.

**Motto for weak infinity categories**

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- $s$ and $t$ are two parallel terms, which can be represented as a type.
- $\sigma$ labels the pasting diagram with (compatible) terms, and can be represented as a substitution.
Identity

Let $t$ be a 1 dimensional term. The identity on $t$ is:

$$\text{coh } (x \xrightarrow{f} y : f \xrightarrow{\sigma} x \rightarrow^* y f)[\sigma]$$

where $\sigma$ maps $f$ to $t$. 
Examples

Identity
Let $t$ be a 1 dimensional term. The identity on $t$ is:

$$\text{coh} \left( x \xrightarrow{f} y : f \xrightarrow{x \to y} f \right)[\sigma]$$

where $\sigma$ maps $f$ to $t$.

1-composition
Let $s : x \to z$ and $t : y \to z$ be terms. Their composite is given by:

$$\text{coh} \left( x \xrightarrow{f} y \xrightarrow{g} z : x \to z \right)[\sigma]$$

where $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(z) = z$, $\sigma(f) = s$, $\sigma(g) = t$. 
Take the context $\Gamma = w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$.

The *associator* is given by:

$$\text{coh} \left( \Gamma : (f \circ g) \circ h \xrightarrow{w \to z} f \circ (g \circ h) \right)[\text{id}]$$
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All weak monoidal categories and all weak 2-categories are equivalent to a strict version of themselves.
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However this is no longer possible at dimensions 3 and higher.
Since full strictification is not possible, we want to do the best possible.
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- Unitors
- Interchangers
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We can strictify:

\begin{tabular}{l|c}
| Strict $\infty$-\textbf{Cat} & \\
|-----------------|---|
| Associators     & ✓  \\
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<table>
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<tr>
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<tr>
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<td>✓</td>
</tr>
<tr>
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</table>
CaTT as we have presented it has no non-trivial equality and no computation.

The idea is to implement a reduction relation that unifies the operations we want to strictify.

By doing this we obtain a type theory for which the models are semistrict categories. Further by showing our reduction is terminating and confluent, we obtain a language for the operations which has decidable type checking and equality.
Current Semistrict Type Theories

- \( \text{CaTT}_{\text{su}} \): Has strict units. Generated by the pruning operation.

- \( \text{CaTT}_{\text{sa}} \): Has strict associators. Generated by the insertion operation.

- \( \text{CaTT}_{\text{sua}} \) (Work in Progress): Combines the previous two theories.
Given two scalars $a, b : id_x \to id_x$, by the Eckmann Hilton argument we have an isomorphism $EH_{f,g} : a \circ_1 b \simeq b \circ_1 a$.

In fact, there are two such isomorphisms, $EH_{a,b}$ and $EH_{b,a}^{-1}$, that need not be themselves isomorphic.

If the whole problem is suspended one dimension higher, then there is a morphism called the syllepsis between these.
Example - Syllepsis

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<tbody>
<tr>
<td>Eckmann-Hilton</td>
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<tr>
<td>Syllepsis</td>
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<td>675</td>
<td>397</td>
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</table>

**Figure:** Coh constructors in Eckmann-Hilton and Syllepsis
Further work

- Finish proving metatheorems for $\text{CaTT}_{\text{sua}}$.
- Equivalence of Theories.
- More semistrict type theories, including one for Simpson-like semistrictness.
- Bridging the gap between CaTT and graphical methods.
