

A Type Theory for Strictly Unital ∞ -Categories

Eric Finster David Reutter Alex Rice Jamie Vicary

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An overview of (globular) infinity categories

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$$x \xrightarrow{f} y$$

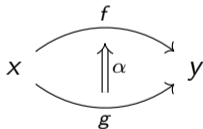
An overview of (globular) infinity categories

Infinity categories contain:

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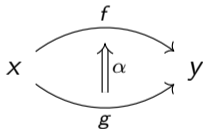
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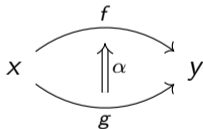
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A diagram showing two 1-arrows, f (top) and g (bottom), both pointing from object x to object y . A 2-arrow, represented by a vertical double arrow labeled α , points from g to f .

- ...

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Compositions:

A diagram showing the composition of two 2-arrows. On the left, 1-arrows f (top) and g (bottom) point from x to y , with a 2-arrow α pointing from g to f . On the right, 1-arrows h (top) and i (bottom) point from y to z , with a 2-arrow β pointing from i to h .

A diagram showing the composition of 1-arrows. 1-arrows f (top) and g (bottom) point from x to y . A 2-arrow α points from g to f . A 1-arrow h points from x to y , positioned between f and g .

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Infinity categories contain:

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$$x \xrightarrow{f} y$$

- 2-arrows:

$$\begin{array}{ccc} & f & \\ x & \curvearrowright & y \\ & \Uparrow \alpha & \\ & g & \end{array}$$

- ...

Our arrows are *Globular*.

Compositions:

$$\begin{array}{ccccc} & f & & h & \\ x & \curvearrowright & y & \curvearrowright & z \\ & \Uparrow \alpha & & \Uparrow \beta & \\ & g & & i & \end{array}$$

$$\begin{array}{ccc} & f & \\ x & \curvearrowright & y \\ & \Uparrow \beta & \\ x & \xrightarrow{h} & y \\ & \Uparrow \alpha & \\ & g & \end{array}$$

Identities:

$$x \xrightarrow{\text{id}_x} x$$

Strict Infinity Categories

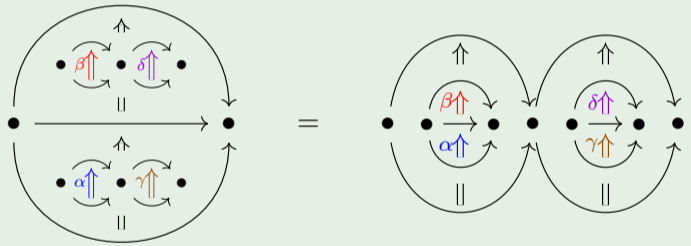
Associativity

$$w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z = w \xrightarrow{f} x \xrightarrow{g \circ h} z$$

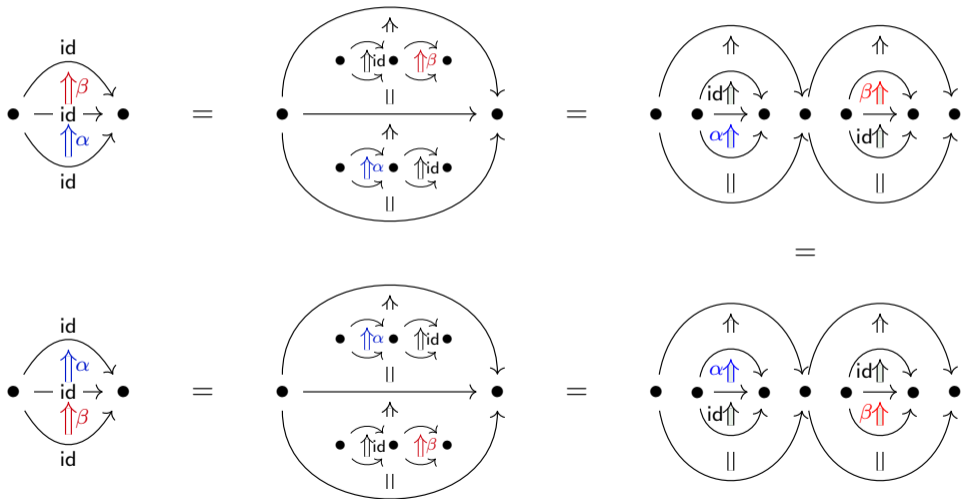
Unitality

The diagram illustrates the unitality property. On the left, a commutative diagram shows the identity element $\text{id}(x)$ and the identity map $\text{id}(\text{id}(x))$ between x and x . The top arrow is $\text{id}(x)$, the bottom arrow is $\text{id}(x)$, and the vertical arrow is $\text{id}(\text{id}(x))$. On the right, a commutative diagram shows the identity element $\text{id}(x)$ and the identity map $\text{id}(\text{id}(x))$ between x and x . The top arrow is $\text{id}(x)$, the bottom arrow is $\text{id}(x)$, and the vertical arrow is $\text{id}(\text{id}(x))$. The two diagrams are connected by an equals sign, indicating that the identity element and the identity map are equal.

Interchange



Example: Eckmann-Hilton



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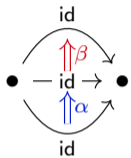
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In a weak higher category, the laws are only required to hold up to isomorphism.

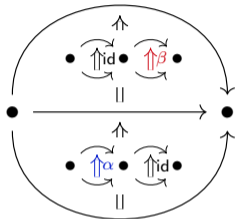
Many examples of higher categories are weak:

- Homotopy groupoids of topological spaces.
- Equality types in HoTT.
- Bicategory of categories and profunctors.

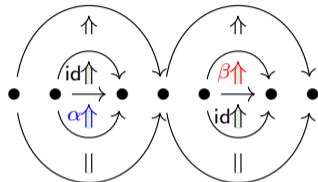
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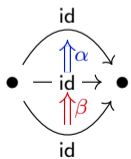
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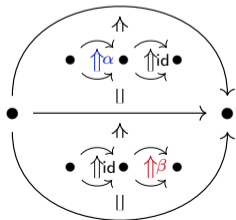
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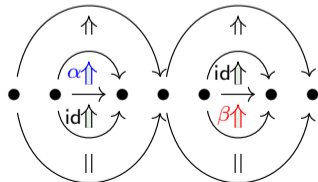
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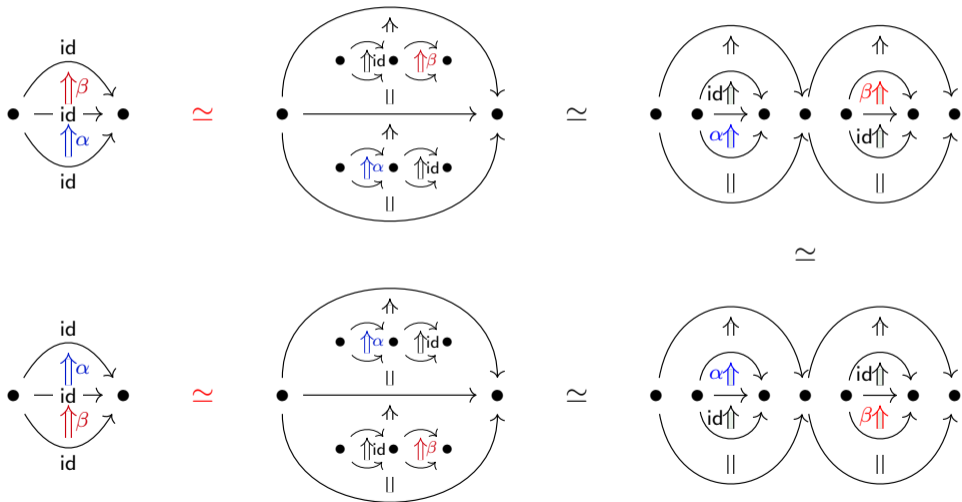
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Example: Eckmann-Hilton



Weakness vs Strictness

Weak ←————→ Strict

Weakness vs Strictness

Weak ←———— Semistrict —————→ Strict

Harder to use

Easier to use

More expressive

Less expressive

Weakness vs Strictness

Weak ←———— Semistrict —————→ Strict

Harder to use

Lack of definitions

Easier to use

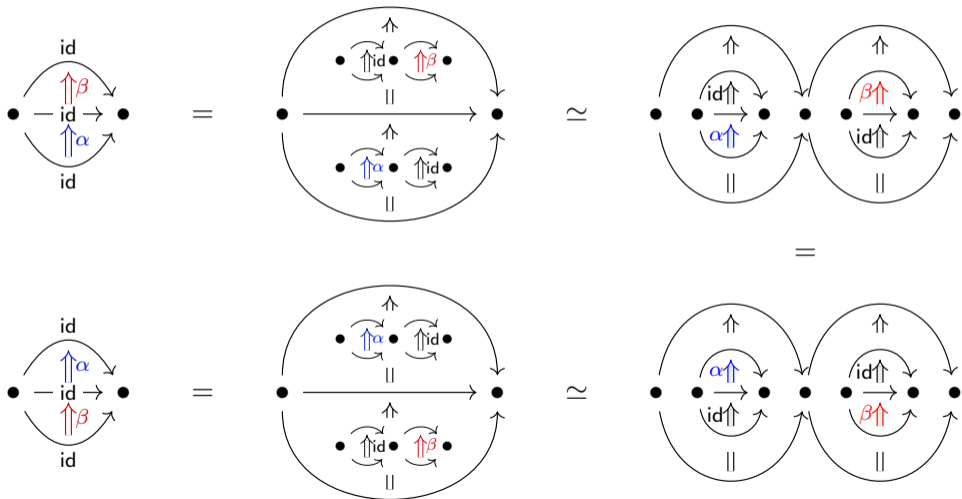
More expressive

Retains expressiveness?

Less expressive

- Catt [1] is a type theory for weak ∞ -categories.
- Its terms are the possible operations in an ∞ -category.
- By adding a definitional equality to Catt, we can unify certain operations.
- Catt_{SU} is a new type theory based on Catt with strict units.

Example: Eckmann-Hilton



```

coh id C (x) : x => x
coh id2 C (x(f)y) : f => f
coh comp C (x(f)y(g)z) : x => z
coh vert C (x(f(a)g(b)h)y) : f => h
coh horiz C (x(f(a)g)y(h(b)k)z) : comp f h => comp g k

```

```

coh swap3 C (x(f(a)g)y(h(b)k)z)
  : vert (horiz a (id2 h)) (horiz (id2 g) b) =>
    vert (horiz (id2 f) b) (horiz a (id2 k))

```

```

let eh {C : Cat} {x :: C} (a :: id x => id x) (b :: id x => id x)
  : [ vert a b => vert b a ]
  = swap3 a b

```

- Equality in Catt_{su} preserves typing.
- Equality is generated by a strongly-terminating, confluent reduction relation.
- Equality and type checking are decidable.
- All terms (of the same dimension) in a disc context are identified.
- Eckmann-Hilton and the Syllepsis have been formalised in Catt_{su} .

- [1] Eric Finster and Samuel Mimram. “A Type-Theoretical Definition of Weak ω Categories”. In: *Proceedings of LICS 2017*. arXiv:1706.02866. 2017.