A Syntax for Strictly Associative and Unital ∞-Categories

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Terms built over *pasting diagrams*.

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$$\Delta := \; \begin{tikzcd} x \arrow[r, bend left=60, "h"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \arrow[r, bend left=60, "i"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Uparrow\beta"] & z \end{tikzcd}$$

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$$s[\![\sigma]\!]$$

$$\sigma := \langle f \mapsto a, g \mapsto (b * c) \rangle$$

$$a * (b * c) := (f * g)[\![\sigma]\!]$$

The Associator

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$$\alpha_{a,b,c} := \mathrm{coh}\,(\Delta : (a * b) * c \to a * (b * c))$$

Semistrictness

The theory described by $\textsc{Catt}$ is *Weak*.

Laws of categories are given by equivalence.

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& \textsc{Catt}_{\mathsf{su}} \arrow[u] & & &
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$$\begin{tikzcd}
u \arrow[r, "c"] & v \arrow[r, "d"] & w \\
x \arrow[u, "f"] \arrow[r, "a"'] \arrow[ur, phantom, "A"] & y \arrow[u, "g"] \arrow[r, "b"'] \arrow[ur, phantom, "B"] & z \arrow[u, "h"']
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\end{tikzcd}$$

$$(\mathrm{id}_a \otimes B) * (A \otimes \mathrm{id}_d) : (a * b) * h \to f * (c * d)$$

$\textsc{Catt}_{\mathsf{sua}}$

$\textsc{Catt}$ has trivial equality.

$\textsc{Catt}_{\mathsf{su}}$ has disc removal, endo-coherence removal, and pruning.

In $\textsc{Catt}_{\mathsf{sua}}$, pruning is replaced by insertion.

$$\textsc{Catt}_{\mathsf{sua}} := \textsc{Catt} + \text{insertion} + \text{disc removal} + \text{endo-coherence removal}$$

Insertion Rule

$$(a * b) * c \;=_{\text{sua}}\; a * b * c \equiv \text{coh}\,(x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \to w)$$

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Recalling $(a * b) * c \equiv (f * g)[\![\langle f \mapsto a, g \mapsto b * c\rangle]\!]$:

$$\begin{tikzcd}
x \arrow[r, "f"] & y \arrow[r, "b*c"] & z & x' \arrow[r, "f'"] & y' \arrow[r, "g'"] & z'
\end{tikzcd}$$

is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

Universal Property of Insertion

$$\mathrm{coh}\ (\Delta : s \to t)[\![\sigma]\!] \quad x \in \Delta \quad x[\![\sigma]\!] \equiv \mathrm{coh}\ (\Theta : u \to v)[\![\tau]\!]$$

$$\begin{tikzcd}
 & \Delta \arrow[dr, bend left, "\sigma"] & \\
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D \arrow[r, "x"] \arrow[d, "\mathrm{coh}\ (\Theta : u \to v)"'] & \Delta \arrow[d, "\kappa"] \arrow[ddr, bend left, "\sigma"] & \\
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$$\mathrm{coh}\,(\Delta : s \to t)[\![\sigma]\!] = \mathrm{coh}\,(\Delta \ll_x \Theta : s[\![\kappa]\!] \to t[\![\kappa]\!])[\![\sigma \ll_x \tau]\!]$$

Normalisation

Equality in $\textsc{Catt}_{\mathsf{sua}}$ is decidable.

$\textsc{Catt}_{\mathsf{sua}}$ has unique normal forms.

Obtained by reduction system.

Termination: Assign *syntactic complexity* to terms.

Confluence: Encode various constructions in Agda.

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Type checking is decidable.

We provide a interpreter which:

- Provides tools for construction terms.
- Type checks terms.
- Reduces terms to $\mathrm{CATT}_{\mathsf{sua}}$ normal form.

Summary

We introduce the type theory $\textsc{Catt}_{\mathsf{sua}}$.

$\textsc{Catt}_{\mathsf{sua}}$ models strictly unital and associative ∞-categories.

$\textsc{Catt}_{\mathsf{sua}}$ terms are simpler than their $\textsc{Catt}$ or $\textsc{Catt}_{\mathsf{su}}$ equivalents.

Normal forms for $\textsc{Catt}_{\mathsf{sua}}$ are obtained via a reduction system:

This reduction system is strongly terminating and confluent.

Try our interpreter for $\textsc{Catt}_{\mathsf{sua}}$:

`https://github.com/alexarice/catt-strict`

Thank you for listening.