A Type Theory for Strictly Associative ∞-Categories

Alex Rice    Eric Finster     Jamie Vicary

SYCO 10
1 Weak Globular Infinity Categories

2 Type Theories for Infinity Categories

3 Strict Associators
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\[ f \xleftarrow{} g \]
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Composition in Globular Sets

Composition of 1 cells

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Composition of 2 cells

Composition along a 1-boundary:

\[ [\bullet \xrightarrow{\beta} \bullet, \bullet \xrightarrow{\alpha} \bullet] \]
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\[ f \rightarrow g \]

Composition of 2 cells

Composition along a 1-boundary:

\[ \alpha \downarrow \quad \beta \downarrow \]

Composition along a 0-boundary:

\[ \alpha \uparrow \quad \beta \uparrow \]
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**Coherence**

- For a 1-cell $f : x \rightarrow y$, there are unitors $\lambda_f : \text{id}_x \circ f \rightarrow f$ and $\rho_f : f \circ \text{id}_y$.
- $\lambda_{\text{id}_x}$ and $\rho_{\text{id}_x}$ are both arrows $\text{id}_x \circ \text{id}_x \rightarrow \text{id}_x$.
- These should be equivalent.
Strict categories are easier to work with while there are more examples of weak categories.
Strictification

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- All weak monoidal categories and all weak 2-categories are equivalent to a strict version of themselves.

- However this is no longer possible at dimensions 3 and higher.
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Semistrictness

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We can strictify:

- Associators
- Unitors
- Interchangers
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- **Contexts**: Generating data of an infinity category.
- **Terms**: Operations in an infinity category.
- **Types**: Source and Target for a term.
- **Substitutions**: A mapping from variables of one context to terms of another.

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$$
\begin{array}{c}
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\alpha \colon f \rightarrow_{x \rightarrow \star y} g
\end{array}
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$$
Contexts consist of a list of pairs of variable names and types.
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**Disc contexts**

For each natural number we can define the *disc context* $D_n$.

$$D_0$$

$$D_1$$

$$D_2$$

$$D_3$$

$$D_2 := x : *, y : *, f : x \to_* y, g : x \to_* y, \alpha : f \to_{x \to_* y} g$$
Composition can be done with the coh constructor.

**coh constructor**

Given:
- A context $\Gamma$ - the shape of the composition,
- A type $A$ in $\Gamma$ - the boundary of the composition,
- A substitution $\sigma : \Gamma \rightarrow \Delta$ - the terms to be composed,

we get a term in $\Delta$:

$$\text{coh} (\Gamma : A)[\sigma]$$

The contexts for which the coh constructor is well typed are called *pasting contexts*.
Suppose we have:

![Diagram](image.png)
Suppose we have:

\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \]

Let \( \Gamma = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \). \( \Gamma \) is a pasting context. Then:

\[ f \cdot g := \text{coh} (\Gamma : x \rightarrow z)[a \mapsto f, \quad b \mapsto g] \]
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\[
(f \cdot g) \cdot h := \text{coh} (\Gamma : x \rightarrow z)[a \mapsto f \cdot g, \\
\hspace{1cm} b \mapsto h]
\]
CaTT as we have presented it has no non-trivial equality and no computation.

The idea is to implement a reduction relation that unifies the operations we want to strictify.

By doing this we obtain a type theory for which the models are semistrict categories.
CaTT_{sa} has a definitional equality based on an operation we call insertion.

1-associator

\[ x \xrightarrow{f} y \xrightarrow{g} z \quad \text{is sent to:} \quad x' \xrightarrow{f'} y' \xrightarrow{g'} z' \]
Components of insertion

\[ \Delta = x \xrightarrow{h} y \xrightarrow{g} z \]
\[ \Theta = x' \xrightarrow{h'} y' \xrightarrow{g'} z' \]

Given \( \sigma : \Delta \rightarrow \Gamma \) and \( \tau : \Theta \rightarrow \Gamma \), we get:

\[ \sigma \ll \alpha \tau : \Delta \ll \alpha \Theta \rightarrow \Gamma \]
Components of insertion

$$\Delta = x \xrightarrow{\beta} g \xrightarrow{\alpha} y \xrightarrow{k} z$$

$$\Theta = x' \xrightarrow{\beta'} g' \xrightarrow{\alpha'} y'$$

$$\Delta \ll_{\alpha} \Theta = x' \xrightarrow{\beta'} h' \xrightarrow{\alpha'} g' \xrightarrow{k} z$$
Components of insertion

\[ \Delta = x \xrightarrow{\beta \uparrow} g \xrightarrow{\alpha \uparrow} y \xrightarrow{k} z \]

\[ \Theta = x' \xrightarrow{\beta' \uparrow} g' \xrightarrow{\alpha' \uparrow} y' \]

\[ \Delta \ll_\alpha \Theta = x' \xrightarrow{\beta' \uparrow} h' \xrightarrow{g'} y' \xrightarrow{k} z \]

\[ \iota : \Theta \rightarrow \Delta \ll_\alpha \Theta \]
Components of insertion

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\[ \iota : \Theta \rightarrow \Delta \ll_{\alpha} \Theta \]

\[ \kappa : \Delta \rightarrow \Delta \ll_{\alpha} \Theta \]
Components of insertion

\[ \Delta = \xrightarrow{h} y \xrightarrow{k} z \]

\[ \Theta = \xrightarrow{h'} y' \]

\[ \Delta \ll \alpha \Theta = \xrightarrow{\alpha} y' \xrightarrow{g'} z \]

\[ \iota : \Theta \to \Delta \ll \alpha \Theta \]

\[ \kappa : \Delta \to \Delta \ll \alpha \Theta \]

Given \( \sigma : \Delta \to \Gamma \) and \( \tau : \Theta \to \Gamma \) we get:

\[ \sigma \ll \alpha \tau : \Delta \ll \alpha \Theta \to \Gamma \]
Insertion also satisfies a *universal property*. Suppose we have coh \((\Delta : A)[\sigma]\) where \(\sigma(\alpha) = \text{coh } (\Theta : B)[\tau]\).
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\[
\begin{array}{c}
D_n \xrightarrow{\alpha} \Delta \\
\downarrow \text{coh} (\Theta:B)[id] \\
\Theta \xrightarrow{\kappa} \Delta \ll \alpha \Theta \\
\downarrow \bar{\imath} \\
\Gamma
\end{array}
\]
Properties of Insertion

Insertion generates a reduction relation for $\text{Catt}_{sa}$:

$$\text{coh } (\Delta : A)[\sigma] \rightsquigarrow \text{coh } (\Delta \ll \alpha \ \Theta : A[\kappa])[\sigma \ll \alpha \ \tau]$$

where $\sigma(\alpha) = \text{coh } (\Delta : B)[\tau]$. 

This reduction has been proven to have the following properties:

- Subject reduction
- Termination
- Confluence
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Finster, Eric and Samuel Mimram. *A Type-Theoretical Definition of Weak $\omega$-Categories.* 2017. DOI: 10.1109/lics.2017.8005124. eprint: 1706.02866.
