A Type Theory for Strictly Associative $\infty$-Categories

Alex Rice    Eric Finster    Jamie Vicary

SYCO 10

UNIVERSITY OF CAMBRIDGE
1. Weak Globular Infinity Categories

2. Type Theories for Infinity Categories

3. Strict Associators
Globular sets are one natural shape of higher categories.
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- A set of objects or 0-cells $G$. 

- For each pair of objects $x, y \in G$, a set of arrows or 1-cells with source $x$ and target $y$.

- For each pair of parallel arrows $f, g$, a set of 2-arrows (or 2-cells) from $f$ to $g$. 

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\[ \xymatrix{ x \ar@/^/[r]^g \ar@/_/[r]_f & y \ar@/^/[u]^-\alpha } \]
Composition in Globular Sets

Composition of 1 cells

Composition along a 1-boundary:

Composition along a 0-boundary:
Composition in Globular Sets

Composition of 1 cells

Composition of 2 cells

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Composition of 1 cells

\[ f \rightarrow g \]

Composition of 2 cells

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Composition along a 0-boundary:
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**Coherence**

- For a 1-cell $f : x \to y$, there are unitors $\lambda_f : \text{id}_x \circ f \to f$ and $\rho_f : f \circ \text{id}_y$.
- $\lambda_{\text{id}_x}$ and $\rho_{\text{id}_x}$ are both arrows $\text{id}_x \circ \text{id}_x \to \text{id}_x$.
- These should be equivalent.
Strictification

- *Strict* categories are easier to work with while there are more examples of *weak* categories.
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Strict categories are easier to work with while there are more examples of weak categories.

All weak monoidal categories and all weak 2-categories are equivalent to a strict version of themselves.

However this is no longer possible at dimensions 3 and higher.
Since full strictification is not possible, we want to do the best possible.
Semistrictness

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We can strictify:

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Associators
Unitors
Interchangers
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CaTT is a type theory for weak infinity categories\(^3\).

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- **Contexts**: Generating data of an infinity category.

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There are 4 pieces of syntax, all defined by mutual induction:

- **Contexts**: Generating data of an infinity category.
- **Terms**: Operations in an infinity category.
- **Types**: Source and Target for a term.
- **Substitutions**: A mapping from variables of one context to terms of another.

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  A term of type \( s \rightarrow_A t \) has source \( s \), target \( t \) and lower dimensional sources and targets given by \( A \).
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\[
\begin{array}{c}
\alpha : f \to_{x \to \star y} g \\
\end{array}
\]
Contexts consist of a list of pairs of variable names and types.
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**Disc contexts**

For each natural number we can define the *disc context* $D_n$.

$D_0$, $D_1$, $D_2$, $D_3$

\[ D_2 := x : *, y : *, f : x \to_* y, g : x \to_* y, \alpha : f \to_{x \to_* y} g \]
Composition can be done with the coh constructor.

**coh constructor**

*Given:*
- A context $\Gamma$ - the shape of the composition,
- A type $A$ in $\Gamma$ - the boundary of the composition,
- A substitution $\sigma : \Gamma \rightarrow \Delta$ - the terms to be composed,

we get a term in $\Delta$:

$$\text{coh} \ (\Gamma : A)[\sigma]$$

The contexts for which the coh constructor is well typed are called *pasting contexts*
Suppose we have:

\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \]
Example composition

Suppose we have:

\[
\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet
\]

Let \( \Gamma = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \). \( \Gamma \) is a pasting context. Then:

\[
f \cdot g := \text{coh } (\Gamma : x \to z)[a \mapsto f, \quad b \mapsto g]
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(f \cdot g) \cdot h := \text{coh}(\Gamma : x \to z)[a \mapsto f \cdot g, \quad b \mapsto h]
\]
CaTT as we have presented it has no non-trivial equality and no computation.

The idea is to implement a reduction relation that unifies the operations we want to strictify.

By doing this we obtain a type theory for which the models are semistrict categories.
CaTT\textsubscript{sa} has a definitional equality based on an operation we call insertion.

1-ass ociator

\[ x \xrightarrow{f} y \xrightarrow{g} z \]

is sent to:

\[ x' \xrightarrow{f'} y' \xrightarrow{g'} z' \]
Components of insertion

\[\Delta = x \xrightarrow{\beta \uparrow} g \xrightarrow{\alpha \uparrow} y \xrightarrow{k} z\]

\[\Theta = x' \xrightarrow{\beta' \uparrow} g' \xrightarrow{\alpha' \uparrow} y'\]
Components of insertion

\[ \Delta = x \xrightarrow{h} g \xrightarrow{\beta} y \xrightarrow{k} z \]

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\[ \Delta \ll \alpha \Theta = x' \xrightarrow{h'} g' \xrightarrow{\beta'} y' \xrightarrow{k} z \]
Components of insertion

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\[ \Delta \preccurlyeq \alpha \Theta = x' \xrightarrow{\beta' \uparrow} g' \xrightarrow{\alpha' \uparrow} y' \xrightarrow{k} z \]

\[ \iota : \Theta \rightarrow \Delta \preccurlyeq \alpha \Theta \]
Components of insertion

\[ \Delta = x \xrightarrow[α↑][β↑] g \rightarrow y \xrightarrow[k] z \]

\[ Θ = x' \xrightarrow[α'↑][β'↑] g' \rightarrow y' \]

\[ \Delta ≪ α \ Θ = x' \xrightarrow[α'↑][β'↑] g' \rightarrow y' \xrightarrow[k] z \]

\[ \iota : Θ \rightarrow \Delta \ll α \ Θ \]

\[ \kappa : \Delta \rightarrow \Delta \ll α \ Θ \]
Components of insertion

$$\Delta = x \xrightarrow{\beta \uparrow} y \xrightarrow{k} z$$

$$\Theta = x' \xrightarrow{\beta' \uparrow} y'$$

$$\Delta \ll\alpha \Theta = x' \xrightarrow{\beta' \uparrow} y' \xrightarrow{k} z$$

$$\iota : \Theta \rightarrow \Delta \ll\alpha \Theta$$

$$\kappa : \Delta \rightarrow \Delta \ll\alpha \Theta$$

Given $\sigma : \Delta \rightarrow \Gamma$ and $\tau : \Theta \rightarrow \Gamma$ we get:

$$\sigma \ll\alpha \tau : \Delta \ll\alpha \Theta \rightarrow \Gamma$$
Insertion also satisfies a *universal property*. Suppose we have $\text{coh} (\Delta : A)[\sigma]$ where $\sigma(\alpha) = \text{coh} (\Theta : B)[\tau]$. 
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![Diagram showing insertion and its properties]
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\[
\begin{array}{c}
D_n \\
\downarrow \text{coh}(\Theta:B)[id] \\
\Theta \\
\downarrow \iota \\
\Delta \ll \alpha \Theta \\
\downarrow \kappa \\
\Delta \\
\downarrow \sigma \\
\Theta \\
\downarrow \tau \\
\Gamma
\end{array}
\]
Insertion also satisfies a *universal property*. Suppose we have $\text{coh} (\Delta : A)[\sigma]$ where $\sigma(\alpha) = \text{coh} (\Theta : B)[\tau]$.
Insertion generates a reduction relation for $\text{Catt}_{sa}$:

$$\text{coh} \ (\Delta : A)[\sigma] \rightsquigarrow \text{coh} \ (\Delta \ll \alpha \ \Theta : A[\kappa])[\sigma \ll \alpha \ \tau]$$

where $\sigma(\alpha) = \text{coh} \ (\Delta : B)[\tau]$. 
Insertion generates a reduction relation for $\text{Catt}_{sa}$:

$$\text{coh } (\Delta : A)[\sigma] \rightsquigarrow \text{coh } (\Delta \ll \alpha \Theta \cdot A[\alpha \Theta])[\sigma \ll \alpha \tau]$$

where $\sigma(\alpha) = \text{coh } (\Delta : B)[\sigma]$.

This reduction has been proven to have the following properties:

- Subject reduction
- Termination
- Confluence
