# A Type Theory for Strictly Associative $\infty$-Categories

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SYCO 10

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# Outline

# Globular Sets

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- For each pair of parallel arrows $f, g$, a set of 2-arrows (or 2-cells) from $f$ to $g$.

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Composition along a 1-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left=80, ""{name=T}] \arrow[r, ""{name=M}] \arrow[r, bend right=80, ""{name=B}] & \bullet
\arrow[from=M, to=T, phantom, "\beta\Uparrow"] \arrow[from=B, to=M, phantom, "\alpha\Uparrow"]
\end{tikzcd}$$

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$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet
\arrow[phantom, from=M, to=U, "\beta\Uparrow"] \arrow[phantom, from=D, to=M, "\alpha\Uparrow"]
\end{tikzcd}$$

Composition along a 0-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left, ""{name=U1, below}] \arrow[r, bend right, ""{name=D1, above}] & \bullet \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2, above}] & \bullet
\arrow[phantom, from=D1, to=U1, "\alpha\Uparrow"] \arrow[phantom, from=D2, to=U2, "\beta\Uparrow"]
\end{tikzcd}$$

# Weak Infinity Categories

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## Coherence

- For a 1-cell $f : x \to y$, there are unitors $\lambda_f : \mathrm{id}_x \circ f \to f$ and $\rho_f : f \circ \mathrm{id}_y$.
- $\lambda_{id_x}$ and $\rho_{\mathrm{id}_x}$ are both arrows $\mathrm{id}_x \circ \mathrm{id}_x \to \mathrm{id}_x$.
- These should be equivalent.

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- However this is no longer possible at dimensions 3 and higher.

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[1] Finster, Reutter, et al., *A Type Theory for Strictly Unital $\infty$-Categories*
[2] Finster, R., and Vicary, *A Type Theory for Strictly Associative Infinity Categories*

CaTT is a type theory for *weak infinity categories*[3].

[3] Finster and Mimram, *A Type-Theoretical Definition of Weak $\omega$-Categories*.

# CaTT

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There are 4 pieces of syntax, all defined by mutual induction:

- Contexts: Generating data of an infinity category.

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- Terms: Operations in an infinity category.
- Types: Source and Target for a term.
- Substitutions: A mapping from variables of one context to terms of another.

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## Contexts

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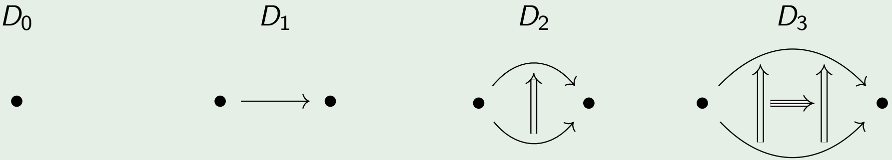
## Disc contexts

For each natural number we can define the *disc context* $D_n$.




$$D_2 := x : \star, y : \star, f : x \to_\star y, g : x \to_\star y, \alpha : f \to_{x \to_\star y} g$$

# Composition in CaTT

Composition can be done with the coh constructor.

**coh constructor**

Given:

- A context $\Gamma$ - the shape of the composition,
- A type $A$ in $\Gamma$ - the boundary of the composition,
- A substitution $\sigma : \Gamma \to \Delta$ - the terms to be composed,

we get a term in $\Delta$:

$$\mathsf{coh}\ (\Gamma : A)[\sigma]$$

The contexts for which the coh constructor is well typed are called *pasting contexts*

# Example composition

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$$\begin{aligned} f \cdot g := \mathsf{coh}\ (\Gamma : x \to z)[a \mapsto f,\\ b \mapsto g] \end{aligned}$$

$$\begin{aligned} (f \cdot g) \cdot h := \mathsf{coh}\ (\Gamma : x \to z)[a \mapsto f \cdot g,\\ b \mapsto h] \end{aligned}$$

# Type theories for semistrict languages

- CaTT as we have presented it has no non-trivial equality and no computation.
- The idea is to implement a reduction relation that unifies the operations we want to strictify.
- By doing this we obtain a type theory for which the models are semistrict categories.

# Insertion

$\text{CaTT}_{\text{sa}}$ has a definitional equality based on an operation we call insertion.

## 1-associator

$$x \xrightarrow{f} y \xrightarrow{g} z \qquad\qquad x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

# Components of insertion

$$\Delta = \begin{tikzcd} x \arrow[r, bend left=50, "h"] \arrow[r, "g" description] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, shift left=3ex, "\beta\Uparrow"] \arrow[r, phantom, shift right=3ex, "\color{red}\alpha\Uparrow"] & y \arrow[r, "k"] & z \end{tikzcd}$$

$$\Theta = \color{red}\begin{tikzcd} x' \arrow[r, bend left=50, "h'"] \arrow[r, "g'" description] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, shift left=3ex, "\beta'\Uparrow"] \arrow[r, phantom, shift right=3ex, "\alpha'\Uparrow"] & y' \end{tikzcd}$$

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$$\Delta = \begin{tikzcd} x \arrow[r, bend left=50, "h"] \arrow[r, "g"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, shift left=2ex, "\beta\Uparrow"] \arrow[r, phantom, shift right=2ex, "\alpha\Uparrow"] & y \arrow[r, "k"] & z \end{tikzcd} \qquad \Theta = \begin{tikzcd} x' \arrow[r, bend left=50, "h'"] \arrow[r, "g'"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, shift left=2ex, "\beta'\Uparrow"] \arrow[r, phantom, shift right=2ex, "\alpha'\Uparrow"] & y' \end{tikzcd}$$

$$\Delta \ll_\alpha \Theta = \begin{tikzcd} x' \arrow[r, bend left=60, "h"] \arrow[r, bend left=20, "h'"] \arrow[r, bend right=20, "g'"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, shift left=3.5ex, "\beta\Uparrow"] \arrow[r, phantom, "\beta'\Uparrow"] \arrow[r, phantom, shift right=3.5ex, "\alpha'\Uparrow"] & y' \arrow[r, "k"] & z \end{tikzcd}$$

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$$\Delta \lll_\alpha \Theta = \begin{tikzcd} x' \arrow[r, bend left=70, "h"] \arrow[r, bend left=25, "h'" description] \arrow[r, bend right=25, "g'" description] \arrow[r, bend right=70, "f'"'] \arrow[r, phantom, shift left=5, "\beta\Uparrow"] \arrow[r, phantom, "\beta'\Uparrow"] \arrow[r, phantom, shift right=5, "\alpha'\Uparrow"] & y' \arrow[r, "k"] & z \end{tikzcd}$$

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$$\Delta \ll\alpha\ \Theta = \begin{tikzcd} x' \arrow[r, bend left=80, "h"] \arrow[r, bend left=25, "h'" description] \arrow[r, bend right=25, "g'" description] \arrow[r, bend right=80, "f'"'] \arrow[r, phantom, shift left=6, "\beta\Uparrow"] \arrow[r, phantom, "\beta'\Uparrow"] \arrow[r, phantom, shift right=6, "\alpha'\Uparrow"] & y' \arrow[r, "k"] & z \end{tikzcd}$$

$$\iota : \Theta \to \Delta \ll\alpha\ \Theta$$

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Given $\sigma : \Delta \to \Gamma$ and $\tau : \Theta \to \Gamma$ we get:

$$\sigma \ll\alpha\ \tau : \Delta \ll\alpha\ \Theta \to \Gamma$$

# Universal Property of Insertion

Insertion also satisfies a *universal property*. Suppose we have $\mathsf{coh}\ (\Delta : A)[\sigma]$ where $\sigma(\alpha) = \mathsf{coh}\ (\Theta : B)[\tau]$.

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\Theta \arrow[dr, bend right, "\tau"'] & \\
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\begin{tikzcd}
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\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"'] & \Delta \ll_\alpha \Theta \arrow[dr, "\sigma \ll_\alpha \tau" description] & \\
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\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"'] & \Delta \ll\alpha\, \Theta \arrow[dr, "\sigma \ll\alpha\, \tau"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & \Gamma
\end{tikzcd}$$

# Properties of Insertion

Insertion generates a reduction relation for $\mathsf{Catt}_{\mathsf{sa}}$:

$$\mathsf{coh}\ (\Delta : A)[\sigma] \rightsquigarrow \mathsf{coh}\ (\Delta \ll_\alpha \Theta : A[\![\kappa]\!])[\sigma \ll_\alpha \tau]$$

where $\sigma(\alpha) = \mathsf{coh}\ (\Delta : B)[\tau]$.

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This reduction has been proven to have the following properties:

- Subject reduction
- Termination
- Confluence

# References


Finster, Eric and Samuel Mimram. *A Type-Theoretical Definition of Weak $\omega$-Categories*. 2017. DOI: 10.1109/lics.2017.8005124. eprint: 1706.02866.

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